

Random greedy triangle-packing beyond the $7/4$ barrier

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Abstract

The random greedy algorithm for constructing a large partial Steiner-Triple-System is defined as follows. Begin with a complete graph on n vertices and proceed to remove the edges of triangles one at a time, where each triangle removed is chosen uniformly at random out of all remaining triangles. This stochastic process terminates once it arrives at a triangle-free graph, and a longstanding open problem is to estimate the final number of edges, or equivalently the time it takes the process to conclude. The intuition that the edge distribution is roughly uniform at all times led to a folklore conjecture that the final number of edges is $n^{3/2+o(1)}$ with high probability, whereas the best known upper bound is $n^{7/4+o(1)}$. It is no coincidence that various methods break precisely at the exponent $7/4$ as it corresponds to the inherent barrier where co-degrees become comparable to the variations in their values that arose earlier in the process.

In this work we significantly improve upon the previous bounds by establishing that w.h.p. the number of edges in the final graph is at most $n^{2-\frac{1}{2\sqrt{2}}+o(1)}$. Our approach relies on a system of martingales used to control key graph parameters, where the crucial new idea is to harness the self-correcting nature of the process in order to control these parameters well beyond the point where their early variation matches the order of their expectation.

1 Introduction

We consider the random greedy algorithm for triangle-packing. This stochastic graph process begins with the graph $G(0)$, set to be the complete graph on vertex set $[n]$, then proceeds to repeatedly remove the edges of randomly chosen triangles (i.e. copies of K_3) from the graph. Namely, letting $G(i)$ denote the graph that remains after i triangles have been removed, the $(i+1)$ -th triangle removed is chosen uniformly at random from the set of all triangles in $G(i)$. The process terminates at a triangle-free graph $G(M)$. In this work we study the random variable M , the number of triangles removed before obtaining a triangle-free graph (or equivalently, the number of edges in the final triangle-free graph, which is $\binom{n}{2} - 3M$).

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This process and its variations play an important role in the history of combinatorics. Rödl [13] proved the Erdős-Hanani conjecture — which posits the existence of large partial Steiner systems, collections of t -sets with the property that no k -set is a subset of more than one set in the collection — in the early 1980's by way of a randomized construction that is now known as the Rödl nibble. This construction is a semi-random variation on the random greedy packing process defined above. It is semi-random in the sense that the desired object is constructed in a sequence of substantial pieces, where the proof of the existence of each piece is an application of the probabilistic method. Such semi-random constructions have been successfully applied to establish various key results in combinatorics (see [1] for an early application of this approach and [3] and [12] for further details). We note in passing that semi-random techniques have been refined to show the existence of partial Steiner systems that are nearly as large as allowed by the simple volume bound, see [11] and [17]. In particular, Alon, Kim and Spencer [2] used such techniques to prove the existence of a collection of edge-disjoint triangles on n vertices that covers all but $O(n^{3/2} \log^{3/2} n)$ of the $\binom{n}{2}$ edges of the complete graph.

Despite the success of the Rödl nibble, the limiting behavior of the random greedy packing process itself remains unknown, even in the special case of triangle packing considered here. Recall that $G(i)$ is the graph remaining after i triangles have been removed. Let $E(i)$ be the edge set of $G(i)$. Note that $|E(i)| = \binom{n}{2} - 3i$ and that $E(M)$ is the number of edges in the triangle-free graph produced by the process. It is widely believed that the graph produced by the random greedy triangle-packing process behaves like the Erdős-Rényi random graph with the same edge density, hence the process should end once its number of remaining edges becomes comparable to the number of triangles in the corresponding Erdős-Rényi random graph (i.e., once $|E(M)|$ matches the order of $(|E(M)|/\binom{n}{2})^3 \binom{n}{3}$). Here and throughout the paper we say that an event holds *with high probability* (w.h.p.) to denote that its probability tends to 1 as $n \rightarrow \infty$.

Conjecture (Folklore). *With high probability $|E(M)| = n^{3/2+o(1)}$.*

Joel Spencer has offered \$200 for a resolution of this question. It was shown by Spencer [15] in 1995, and independently by Rödl and Thoma [13] in 1996, that $|E(M)| = o(n^2)$ w.h.p. Grable [10] improved this bound to $|E(M)| \leq n^{11/6+o(1)}$ via an adaptation of the Rödl nibble method and further sketched how similar arguments using more delicate calculations should extend this to a bound of $n^{7/4+o(1)}$ w.h.p. Wormald [18] later demonstrated how the differential equation method can also give nontrivial bounds for this problem (as well as generalizations of it), and namely that $|E(M)| \leq n^{2-\frac{1}{57}+o(1)}$. Finally, in a companion paper [5] that introduced a differential equation approach to this process exploiting its self-correction nature, the foundations of the present work, the authors gave a short proof that $|E(M)| = O(n^{7/4+o(1)})$ w.h.p.

It is important to note that the point at which there are roughly $n^{7/4}$ remaining edges is a natural barrier in the analysis of this process. To illustrate this, notice that if the $(i+1)$ -st triangle taken is abc then the change in the number of triangles in the graph once abc is removed is simply $-|N_{a,b}(i)| - |N_{a,c}(i)| - |N_{b,c}(i)| + 2$, where $N_{u,v}(i)$ denotes the common neighborhood of the vertices $u, v \in [n]$ in the graph $G(i)$. Hence, a natural prerequisite to analyzing this process is the understanding of the co-degrees $|N_{u,v}|$ for all u, v . Suppose for the sake of this discussion that early in the evolution of the process $G(i)$ closely resembles the random graph with the same number of edges; that is, suppose $G(i)$ is roughly the same as $G_{n,p}$ where $p = p(i) = 1 - 3i/\binom{n}{2}$. If

this is the case when p is close to $1/2$ (i.e. i is nearly $\binom{n}{2}/6$) then we expect the $|N_{u,v}|$'s to be close to $n/4$ with variations as large as \sqrt{n} . If these variations in co-degrees persist to the point where $p = n^{-1/4}$ (that is, i roughly $\binom{n}{2}/3 - n^{7/4}$), where we expect the $|N_{u,v}|$'s themselves to be roughly $n^{1/2}$, then these variations would be as large as their average value. Once this happens all control over co-degrees is lost, e.g. one could have all co-degrees 0 with non-vanishing probability, or half of the co-degrees 0 and the other half around $n^{1/2}$, etc. In any case, if the variations in $|N_{u,v}|$ that develop early in the process are not somehow dealt with, one would expect the analysis to break down once $n^{7/4}$ edges remain. Perhaps this is the reason that Wormald [18], who also treated this process with the differential equation method, stated that “some non-trivial modification would be required to equal or better Grable’s result.”

In this work we exploit the self-correcting nature of the process in a system of carefully constructed martingales which allows us to tighten the control over key graph properties over time and overcome the variations in their values that arise early in the process. Our main result is an upper bound on $|E(M)|$ that is significantly better than $n^{7/4}$.

Theorem 1. *Consider the random greedy algorithm for triangle-packing on n vertices. Let M be the number of steps it takes the algorithm to terminate and let $E(M)$ be the edges of the resulting triangle-free graph. Then with high probability $|E(M)| = O(n^{2-\frac{1}{2\sqrt{2}}} \log^4 n)$.*

A key feature of our proof of Theorem 1 is an estimate for $|N_{u,v}|$ in which the variation *decreases* as the process evolves. We stress that estimates for random graph processes with this property are not commonly obtained by martingale arguments or the differential equation method.

The remainder of the paper is organized as follows. In the next section we discuss our analysis of this process in more detail, listing the random variables that we track and the estimates on them that we are able to prove. The proof of our main result, Theorem 2.1, follows in Section 3. Theorem 1 follows directly from Theorem 2.1.

2 Self-correcting Estimates

Let (\mathcal{F}_i) be the filtration given by the underlying process. We note in passing that our probability space is the set of all maximal sequences of edge-disjoint triangles on vertex set $[n]$ with probability measure given by the uniform random choice at each step.

For $u, v, w \in [n]$ define $N_u = N_u(i) = \{x \in [n] : xu \in E(i)\}$, let $N_{u,v} = N_u \cap N_v$ and let $N_{u,v,w} = N_u \cap N_v \cap N_w$. Our main interest is in tracking the number of triangles in $G(i)$ and the variables

$$Y_{u,v}(i) = |N_{u,v}(i)| = |\{x \in [n] : xu \in E(i), xv \in E(i)\}|.$$

In the course of our argument we will also need to consider the variables

$$Y_u(i) = |N_u(i)| \quad \text{and} \quad Y_{u,v,w}(i) = |N_{u,v,w}(i)|.$$

We begin by writing the one-step expected changes in our main variables of interest. For any random variable W let ΔW be the one-step change $\Delta W = W(i+1) - W(i)$. Let $Q(i)$ be the

number of triangles in $G(i)$. We have

$$\mathbb{E}[\Delta Y_{u,v} \mid \mathcal{F}_i] = - \sum_{x \in N_{u,v}} \frac{Y_{u,x} + Y_{v,x} - \mathbb{1}_{\{uv \in E\}}}{Q}, \quad (2.1)$$

$$\mathbb{E}[\Delta Q \mid \mathcal{F}_i] = - \sum_{xyz \in Q} \frac{Y_{x,y} + Y_{x,z} + Y_{y,z} - 2}{Q}. \quad (2.2)$$

We use these one-step expected changes to relate the random variables to functions of a continuous ‘time’ (following the approach to the differential equation method introduced in [4]). We choose the time-scaling $t = t(i) = i/n^2$. Following the convention established in the Introduction we set

$$p = p(i, n) = 1 - \frac{6i}{n^2} = 1 - 6t. \quad (2.3)$$

Note that p can now be viewed as either a function of i or the continuous time t ; we pass between these interpretations without further comment throughout the paper. Now, these choices yield the trajectories $Y_{u,v}(i) \approx y(t)n$ and $Q(i) \approx q(t)n$ where we set

$$y(t) = p^2(t) \quad \text{and} \quad q(t) = p^3(t)/6.$$

We can arrive at these equations by either deriving them from (2.1) and (2.2) and the assumption that the one-step changes in the trajectory are equal to the expected one-step change in the corresponding random variable or by appealing to our $G_{n,p}$ intuition. The companion paper [5] uses these variables alone to establish a bound of $O(n^{7/4} \log^{5/4} n)$ on the number of edges that survive to the conclusion of the algorithm.

In order to achieve better precision, we introduce additional variables with the central goal of establishing an estimate for $Y_{u,v}$ with variation that decreases as the process evolves. (For applications of the differential equation method that exploit this kind of ‘self-correcting’ phenomenon, see [7] and [16].) We would like to add random variables to our collection that will give us better control on the expression in the numerator of (2.1), the one-step expected change in $Y_{u,v}$. To this end we take a closer look at this expression. We have

$$\sum_{x \in N_{u,v}} (Y_{u,x} + Y_{v,x} - \mathbb{1}_{\{uv \in E\}}) = 4P_{u,v} + R_{u,v} + S_{u,v} + Y_{u,v} \mathbb{1}_{\{uv \in E\}} \quad (2.4)$$

where

$$\begin{aligned} R_{u,v} &= \left| \{xy \in E(i) : x \in N_{u,v}, y \in N_u \setminus N_{u,v} \text{ and } y \neq v\} \right|, \\ S_{u,v} &= \left| \{xy \in E(i) : x \in N_{u,v}, y \in N_v \setminus N_{u,v} \text{ and } y \neq u\} \right|, \\ P_{u,v} &= \left| E(i) \cap \binom{N_{u,v}}{2} \right|. \end{aligned}$$

We will use appropriate martingales to track each of these variables for every pair of vertices uv . Now, we expect these variables to be correlated with Y_u, Y_v and $Y_{u,v}$. Indeed, we should have

$$R_{u,v} \approx p(Y_u - Y_{u,v})Y_{u,v}, \quad S_{u,v} \approx p(Y_v - Y_{u,v})Y_{u,v}, \quad P_{u,v} \approx p \binom{Y_{u,v}}{2}.$$

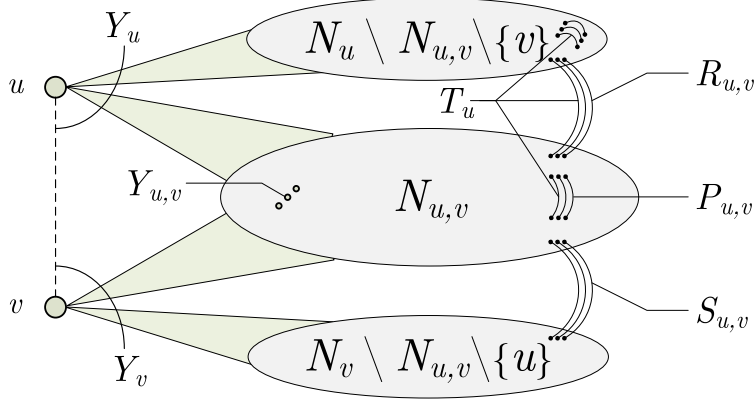


Figure 1: The ensemble of variables $Y_{u,v}, R_{u,v}, S_{u,v}, P_{u,v}, Y_u, T_u$.

Dropping the lower-order term that is linear in $Y_{u,v}$ now suggests the following approximation:

$$4P_{u,v} + R_{u,v} + S_{u,v} \approx p Y_{u,v}(Y_u + Y_v).$$

Note that this expression is in a form that should provide self-correction in our estimate for $Y_{u,v}$: If $Y_{u,v}$ is large compared with its expected trajectory then this term will be large and (as this term is negated in the expected one-step change) there is a ‘drift’ that pulls $Y_{u,v}$ back toward the expected trajectory.

Note further that our expression for $4P_{u,v} + R_{u,v} + S_{u,v}$ involves Y_u and Y_v . So, for this approach to be useful we will need some control on vertex degrees. We achieve this control using the same mechanism we are using to control co-degrees. Namely, we consider the expression in the numerator of the expected change in Y_u ,

$$\mathbb{E}[\Delta Y_u \mid \mathcal{F}_i] = -\frac{1}{Q} \sum_{x \in Y_u} Y_{u,v} = -\frac{2T_u}{Q}$$

where

$$T_u = \left| \{xy \in E(i) : x, y \in N_u\} \right|,$$

and track this quantity for every vertex u . This quantity should be correlated with Y_u ; that is, we should have

$$T_u \approx p \binom{Y_u}{2}.$$

We arrive at the ensemble of variables $Q, Y_{u,v}, R_{u,v}, S_{u,v}, P_{u,v}, Y_u, T_u$ and $Y_{u,v,w}$ (it turns out that some control on the common neighborhoods of triples is needed for our martingale arguments), as illustrated in Figure 1. We are ready to state our main result, which establishes concentration for the entire ensemble of variables and in particular provides us with the level control over Q required to deduce Theorem 1. Here and in what follows the notation $A = B \pm C$ is short for $A \in [B - C, B + C]$.

Theorem 2.1. Let $p(i, n) = 1 - 6i/n^2$ and set

$$\gamma = 2 - \sqrt{2}, \quad \hat{\gamma} = \hat{\gamma}(n) = \gamma - \frac{6}{\log n} \quad \text{and} \quad \Phi = \Phi(\Upsilon, i, n) = e^{\Upsilon(1-p)} \log^2 n.$$

Then there exist absolute constants $\alpha, \beta, \delta, \kappa, \mu, \Upsilon > 0$ such that with high probability

$$Q = n^3 p^3 / 6 \pm \alpha^2 n^2 p^{2\hat{\gamma}-1} \Phi^2 \quad (2.5)$$

$$Y_{u,v} = np^2 \pm \alpha n^{1/2} p^{\hat{\gamma}} \Phi \quad (2.6)$$

$$R_{u,v} = pY_{u,v}(Y_u - Y_{u,v}) \pm \beta n^{3/2} p^{2+\hat{\gamma}} \Phi \quad (2.7)$$

$$S_{u,v} = pY_{u,v}(Y_v - Y_{u,v}) \pm \beta n^{3/2} p^{2+\hat{\gamma}} \Phi \quad (2.8)$$

$$P_{u,v} = pY_{u,v}^2 / 2 \pm \delta n^{3/2} p^{3+\hat{\gamma}} \Phi \quad (2.9)$$

$$Y_u = np \pm \kappa n^{1/2} p^{\hat{\gamma}-1} \Phi \quad (2.10)$$

$$T_u = pY_u^2 / 2 \pm \mu n^{3/2} p^{1+\hat{\gamma}} \Phi \quad (2.11)$$

$$Y_{u,v,w} = np^3 \pm 2 \max \left\{ (np^3 \log^5 n)^{1/2}, \log^5 n \right\} \quad (2.12)$$

for all u, v, w and throughout the time-steps $0 \leq i \leq \left[1 - (6\alpha^2 e^{2\Upsilon} \frac{\log^{10} n}{n})^{1/(4-2\hat{\gamma})} \right] n^2 / 6$.

To deduce Theorem 1, let p^* be the lowest value of p for which Theorem 2.1 holds and note that

$$p^* = \left(6\alpha^2 e^{2\Upsilon} \frac{\log^{10} n}{n} \right)^{1/(4-2\hat{\gamma})} = O \left(n^{-1/(2\sqrt{2})} \log^{5/\sqrt{2}} n \right) \quad (2.13)$$

since $4 - 2\hat{\gamma} = 2\sqrt{2} + O(1/\log n)$. In particular, $p = p^*$ satisfies $n^3 p^3 / 6 > \alpha^2 n^2 p^{2\hat{\gamma}-1} \Phi^2$ since $\Phi = e^{\Upsilon(1-p)} \log^2 n \leq e^{\Upsilon} \log^2 n$. It thus follows that $Q > 0$ w.h.p. due to (2.5) and it remains to recover the number of edges corresponding to p^* . Recalling (2.3) we have

$$|E(i)| = \binom{n}{2} - 3i = \binom{n}{2} - \frac{1}{2}(1 - p^*)n^2 = \frac{1}{2}(n^2 p^* - n)$$

and the desired result follows from (2.13) with room to spare in the power of the logarithmic factor.

We prove Theorem 2.1 in the following section by applying martingale arguments to random variables that track the differences between the random variables we are interested in and the variables they should follow. Note that we establish some form of self-correction for every variable in this ensemble, with the notable exception of Y_u .

The authors suspect that the methods introduced in this paper can be further developed to achieve better high probability upper bounds on $|E(M)|$. This might be achieved by expanding the ensemble of random variables (perhaps using ensembles of generalized extension-counting variables, which is the approach taken in the recent analysis of the H -free process [6]). However, it seems that a nontrivial modification would be needed to prove the conjectured bound $|E(M)| = n^{3/2+o(1)}$.

For notational convenience we set

$$\Lambda = \frac{1}{\log^2 n}.$$

Note that while Theorem 2.1 applies, estimates (2.6)-(2.11) and (2.5) can each be written as a main term times $(1 + O(1/\log^2 n)) = (1 + O(\Lambda))$. Throughout the paper we will use a convention

whereby all Greek letters are universal constants. We do not replace any of the constants (including the pivotal γ) with their actual values. This is done in the interest of understanding the role these constants play in the calculations; it turns out that these constants must be balanced in a fairly delicate way. We observe that these constants can take the actual values

$$\begin{aligned} \alpha &= \frac{1}{2} & \beta &= \frac{\alpha}{2} & \delta &= \frac{\alpha}{4 - 2\gamma} \\ \kappa &= \frac{(1 - \gamma)\alpha}{2} & \mu &= \frac{\alpha}{4} & \Upsilon &= 20. \end{aligned}$$

The key conditions these constants must satisfy are (3.19), (3.22), (3.24), (3.28) and (3.29). The only condition on Υ is that it is sufficiently large.

3 Proof of Theorem 2.1

Define p^* as in (2.13) and let $i^* = \frac{1}{6}(1 - p^*)n^2$ be the analogous round. Let \mathcal{G}_i be the event that all estimates in Theorem 2.1 hold for the first i steps of the process.

For each variable and each bound (upper and lower) in Theorem 2.1 we define a *critical interval*. This interval has one end at the bound we are trying to maintain and the other end slightly closer to the expected trajectory of the random variable. If one of the estimates of Theorem 2.1 is violated then the corresponding random variable ‘crosses’ a critical interval. We bound the probability of each such event using a martingale argument, introducing a separate supermartingale for each variable and bound of interest and for each step in which the variable could *enter* the critical interval. Theorem 2.1 then follows from the union bound (note that the number of supermartingales we consider is bounded by a polynomial in n). We restrict our attention to these critical intervals because the expected one-step changes in our random variables each have a ‘drift’ term that pushes a wayward variable back toward the expected trajectory. By only considering the critical intervals we make full use of these terms: This is the mechanism we are using to establish self-correcting estimates. For an application of this idea in a setting with fewer variables, see [5].

Let the stopping time τ be the minimum of i^* and the smallest index i such that \mathcal{G}_i does not hold. Consider an event \mathcal{E} of the form $X(i) \leq x(t)$ for all $i \leq i^*$ where we assume that $X(i)$ is a random variable and $x(t)$ is not. Note that every bound in (2.5)-(2.12) can be written in this form; that is, the event $\{\tau = i^*\}$ can be written

$$\{\tau = i^*\} = \bigcap_{\ell \in \mathcal{I}} \mathcal{E}_\ell$$

where $|\mathcal{I}|$ is polynomial in n and each event \mathcal{E}_ℓ is of the form $X(i) \leq x(t)$ for all $i \leq i^*$. For each such event \mathcal{E} we introduce a critical interval of the form $I_\mathcal{E}(t) = (x(t) - w(t), x(t))$ where $w(t) = o(x(t))$. Consider a *fixed step* i_0 , which we view as a step at which $X(i)$ might enter the critical interval $I_\mathcal{E}$. Set $t_0 = i_0/n^2$. Define the stopping time $\tau_{\mathcal{E}, i_0}$ to be the minimum of $\max\{i_0, \tau\}$ and the smallest $i \geq i_0$ such that $X(i) \notin I_\mathcal{E}$. Note that if $X(i_0) \notin I_\mathcal{E}(t_0)$ then we have $\tau_{\mathcal{E}, i_0} = i_0$. Thus this stopping time is (formally) well-defined on the full probability space (n.b. we only make use of this stopping time when $X(i_0)$ is in the critical interval and $X(i_0 - 1)$ is not). We now establish a bound $B(i)$ on the one-step change in $X(i)$ conditioned on \mathcal{G}_i . This bound is far less than the width $w(t)$ of

the critical interval. Given a particular event \mathcal{E} and starting step i_0 , we work with the sequence of random variables

$$Z_{\mathcal{E},i_0}(i) = Z(i) = \begin{cases} X(i) - x(t) & \text{if } i_0 \leq i \leq \tau_{\mathcal{E},i_0} \\ Z(i-1) & \text{otherwise.} \end{cases}$$

Note that if $X(i_0 - 1)$ is not in the critical interval (and $i_0 < \tau$) then $Z(i_0) < -w(t_0) + B(i_0)$. Therefore, in the event \mathcal{E}^c there are steps $i_0 < j \leq i^*$ such that $Z(i_0) < -w(t_0) + B(i_0)$ and $Z(j) \geq 0$. However, our stopping time τ stops all of these sequences as soon as any of our conditions (2.5)-(2.12) are violated. So, we have

$$\begin{aligned} \{\tau < i^*\} &\subseteq \bigcup_{\ell \in \mathcal{I}} \bigcup_{1 \leq i_0 < i^*} \{Z(\tau) - Z(i_0) > w(t_0) - B(i_0)\} \\ &= \bigcup_{\ell \in \mathcal{I}} \bigcup_{1 \leq i_0 < i^*} \{Z(i^*) - Z(i_0) > w(t_0) - B(i_0)\}. \end{aligned}$$

It remains to bound the probability of each event in this union. This is done for each of the bounds (2.5)-(2.12) in turn in Sections 3.1 – 3.7. In order to bound the probability of these events we will apply the following inequality due to Freedman [9], which was originally stated for martingales yet its proof extends essentially unmodified to supermartingales.

Theorem 3.1 ([9], Thm 1.6). *Let (S_0, S_1, \dots) be a supermartingale w.r.t. a filter (\mathcal{F}_i) . Suppose that $S_{i+1} - S_i \leq B$ for all i , and write $V_t = \sum_{i=0}^{t-1} \mathbb{E}[(S_{i+1} - S_i)^2 \mid \mathcal{F}_i]$. Then for any $s, v > 0$*

$$\mathbb{P}(\{S_t \geq S_0 + s, V_t \leq v\} \text{ for some } t) \leq \exp \left\{ - \frac{s^2}{2(v + Bs)} \right\}.$$

Our applications of this inequality will each have two parts: a careful calculation that establishes a martingale condition and a coarser argument that provides bounds on both the one-step changes and the second moment of the one-step changes of these variables. We emphasize that our carefully chosen stopping times allow us to assume that the event \mathcal{G}_i holds throughout these calculations. This is henceforth assumed without further comment.

3.1 Edges between a co-neighborhood and an exclusive neighborhood $(R_{u,v}, S_{u,v})$

In this section we establish concentration for $R_{u,v}, S_{u,v}$ (by symmetry it suffices to consider $R_{u,v}$) throughout the process via a delicate analysis of the behavior near the tails of their distributions.

We begin with a calculation of the one-step (conditional) expected change in $R_{u,v}$. To simplify the exposition we will write $xy \in R_{u,v}$ to denote that the edge xy is counted by the variable $R_{u,v}$, and similarly for $S_{u,v}, P_{u,v}, T_u$. There are 7 types of triangles that contribute to this expected change (see Figure 2 for an illustration):

1. Triangles vxy where $x \in N_{u,v}$ and $y \in N_v \setminus N_{u,v}$ and $y \neq u$. For a fixed $x \in N_{u,v}$ there are $Y_{v,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}}$ such triangles and selection of one of these triangles moves x from $N_{u,v}$ to $N_u \setminus N_{u,v}$ and thereby decreases $R_{u,v}$ by $Y_{u,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}}$ (due to the neighbors of x in the exclusive neighborhood of u) and increases $R_{u,v}$ by $Y_{u,v,x}$ (due to the neighbors of x

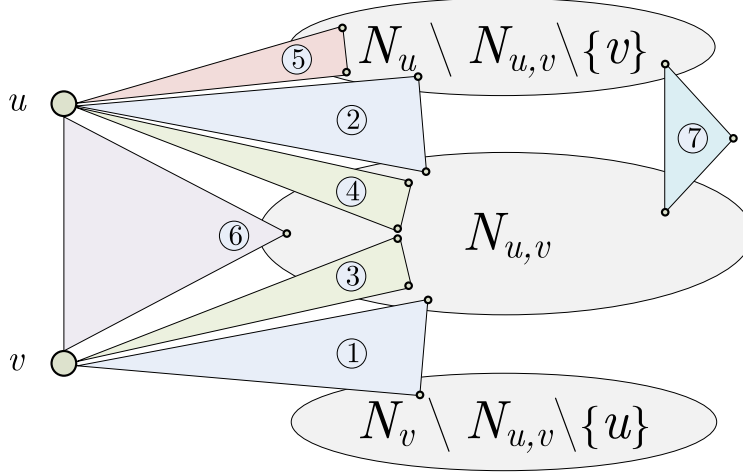


Figure 2: The types of triangles contributing to the change in $R_{u,v}$.

in $N_{u,v}$). This results in a contribution of

$$\begin{aligned}
& - \sum_{x \in N_{u,v}} \frac{Y_{v,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}}}{Q} ((Y_{u,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}}) - Y_{u,v,x}) \\
& = - \sum_{x \in N_{u,v}} \frac{Y_{v,x} - Y_{u,v,x}}{Q} (Y_{u,x} - 2Y_{u,v,x}) + O\left(\frac{p}{n}\right), \tag{3.1}
\end{aligned}$$

where in the last equality we absorbed the indicator variables into an $O(p/n)$ term based on the estimates of Theorem 2.1 that hold up to the stopping time τ .

2. Triangles uxy where $x \in N_{u,v}$ and $y \in N_u \setminus N_{u,v}$, i.e. triangles given by $xy \in R_{u,v}$. The selection of such a triangle removes y from N_u while moving x from $N_{u,v}$ to $N_v \setminus N_{u,v}$. Noting that there are $Y_{u,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}}$ edges of the form $xy \in R_{u,v}$ for each $x \in N_{u,v}$ while there are $Y_{u,v,y}$ edges $xy \in R_{u,v}$ for each $y \in N_u \setminus N_{u,v}$, selecting the triangle uxy decreases $R_{u,v}$ by $Y_{u,v,y} + Y_{u,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}} - 1$. Overall this results in a contribution of

$$- \sum_{x \in N_{u,v}} \frac{(Y_{u,x} - Y_{u,v,x})^2}{Q} - \sum_{y \in N_u \setminus N_{u,v}} \frac{Y_{u,v,y}^2}{Q} + O\left(\frac{p}{n}\right), \tag{3.2}$$

where the indicator terms were again absorbed into the $O(p/n)$ term.

3. Triangles vxy where $xy \in P_{u,v}$. Choosing such a triangle moves x, y from $N_{u,v}$ to $N_u \setminus N_{u,v}$ and thus decreases $R_{u,v}$ by $(Y_{u,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}}) + (Y_{u,y} - Y_{u,v,y} - \mathbb{1}_{\{uv \in E\}})$ and increases $R_{u,v}$ by $Y_{u,v,x} + Y_{u,v,y} - 2$. This contributes

$$\begin{aligned}
& - \sum_{xy \in P_{u,v}} \frac{1}{Q} (Y_{u,x} - 2Y_{u,v,x} + Y_{u,y} - 2Y_{u,v,y}) + O\left(\frac{p^2}{n}\right) \\
& = - \sum_{x \in N_{u,v}} \frac{Y_{u,v,x}}{Q} (Y_{u,x} - 2Y_{u,v,x}) + O\left(\frac{p^2}{n}\right). \tag{3.3}
\end{aligned}$$

4. Triangles uxy where $xy \in P_{u,v}$. Selection of such a triangle moves x, y from $N_{u,v}$ to $N_v \setminus N_{u,v}$ and so decreases $R_{u,v}$ by $(Y_{u,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}}) + (Y_{u,y} - Y_{u,v,y} - \mathbb{1}_{\{uv \in E\}})$ (similar to the previous case only without introducing any new edges to $R_{u,v}$). This amounts to

$$\begin{aligned} & - \sum_{xy \in P_{u,v}} \frac{1}{Q} (Y_{u,x} - Y_{u,v,x} + Y_{u,y} - Y_{u,v,y}) + O\left(\frac{p^2}{n}\right) \\ & = - \sum_{x \in N_{u,v}} \frac{Y_{u,v,x}}{Q} (Y_{u,x} - Y_{u,v,x}) + O\left(\frac{p^2}{n}\right). \end{aligned} \quad (3.4)$$

5. Triangles uxy where xy is in the set of edges induced by $N_u \setminus N_{u,v}$, i.e. $xy \in T_u \setminus (R_{u,v} \cup P_{u,v})$. Each such triangle decreases $R_{u,v}$ by $Y_{u,v,x} + Y_{u,v,y}$. This results in a contribution of

$$- \sum_{xy \in T_u \setminus (R_{u,v} \cup P_{u,v})} \frac{1}{Q} (Y_{u,v,x} + Y_{u,v,y}) = - \sum_{x \in N_u \setminus N_{u,v}} \frac{Y_{u,x} - Y_{u,v,x}}{Q} Y_{u,v,x}. \quad (3.5)$$

6. Triangles uvx for $x \in N_{u,v}$ contribute

$$- \mathbb{1}_{\{uv \in E\}} \sum_{x \in N_{u,v}} \frac{Y_{u,x} - Y_{u,v,x} - 1}{Q} = - \frac{R_{u,v} \mathbb{1}_{\{uv \in E\}}}{Q} = O\left(\frac{p}{n}\right). \quad (3.6)$$

7. Triangles xyw where $x \in N_{u,v}$ and $y \in N_u \setminus N_{u,v}$ and $w \neq u$. We note in passing that this is the only type of triangle whose selection impacts $R_{u,v}$ while changing neither $Y_u - Y_{u,v}$ nor $Y_{u,v}$. For a fixed $xy \in R_{u,v}$ there are $Y_{x,y} - 1$ such triangles and this results in a total contribution of

$$- \sum_{xy \in R_{u,v}} \frac{Y_{x,y}}{Q} + O\left(\frac{p}{n}\right) \quad (3.7)$$

to the expected change in $R_{u,v}$. If we simply apply our bounds on $Y_{x,y}$ to the expression (3.7) we would get an undesirable error. A more careful approach will give better bounds. Namely, we rewrite this expression by summing over $x \in N_{u,v}$ and using our error bounds on $S_{u,x}$ and $P_{u,x}$. This should give a better estimate as $S_{u,x}$ and $P_{u,x}$ ‘aggregate’ many terms $Y_{x,y}$ and should therefore produce better error bounds, and indeed this proves to be a crucial choice.

There are two types of triangle xyw in this category depending on whether or not uw is an edge. Consider first the triangles where uw is not an edge. Observe that an edge yw which is in $S_{u,x}$ due to having $y \in N_{u,x}$ and $w \in N_x \setminus N_{u,x}$ determines a triangle of this type so long as $y \neq v$ and $y \notin N_v$. The requirement $y \neq v$ excluded $S_{u,v}$ edges from being yw candidates provided that $uv \in E$, whereas edges excluded due to $y \in N_v$ correspond to having $x, y \in N_{u,v}$ with $xy \in E$ and $w \in N_{x,y} \setminus (N_{u,x,y} \cup \{u\})$. Therefore, the number of triangles of this type is

$$\begin{aligned} & \sum_{x \in N_{u,v}} S_{u,x} - \mathbb{1}_{\{uv \in E\}} S_{u,v} - \sum_{xy \in P_{u,v}} 2 [Y_{x,y} - 2 - Y_{u,x,y} + \mathbb{1}_{\{uv \in E\}}] \\ & = \sum_{x \in N_{u,v}} \left[p Y_{u,x} (Y_x - Y_{u,x}) \pm \beta n^{3/2} p^{2+\hat{\gamma}} \Phi \right] - \sum_{xy \in P_{u,v}} 2 [Y_{x,y} - Y_{u,x,y}] + O(n^2 p^4) \\ & = \sum_{x \in N_{u,v}} p Y_{u,x} (Y_x - Y_{u,x}) - \sum_{xy \in P_{u,v}} 2 [Y_{x,y} - Y_{u,x,y}] \pm (1 + O(\Lambda)) \beta n^{5/2} p^{4+\hat{\gamma}} \Phi, \end{aligned} \quad (3.8)$$

where in the last equality we absorbed the $O(n^2 p^4)$ error term using the fact that $\Lambda = \log^{-2} n$ is $O(n^{1/2} p^{\hat{\gamma}})$ (with room to spare).

We now turn to treat triangles where $uw \in E$, that is, triangles xyw which either have $w, y \in N_u \setminus N_{u,v}$ and $x \in N_{u,v}$ or alternatively $y \in N_u \setminus N_{u,v}$ and $w, x \in N_{u,v}$. Note that the removal of such a triangle will decrease $R_{u,v}$ by 2. The effect of removing these triangles can be counted by $\sum_{x \in N_{u,v}} P_{u,x}$ using a delicate double-count: In the case where $w \in N_u \setminus N_{u,v}$ the aforementioned sum counts the triangle xyw once, whereas this triangle is counted twice if $w \in N_{u,v}$. Furthermore, this sum includes pairs $yw \in P_{u,x}$ with $w, y \in N_{u,v}$, which do not affect $R_{u,v}$ at all. Weighing in the number of triangles xyw as above together with their effect on $R_{u,v}$ now totals to taking $2P_{u,x}$, negating 1 per triangle with $w \in N_{u,v}$ and another 1 per triangle with $w \in N_{u,v,x}$ (the last case is counted twice by the edge yw with $w, y \in N_{u,v}$ and hence cancels away the contribution of this triangle). Altogether this sums to

$$\begin{aligned}
& \sum_{x \in N_{u,v}} \left[2(P_{u,x} - \mathbb{1}_{\{uv \in E\}} Y_{u,v,x}) - \sum_{w \in N_{u,v,x}} (|N_{u,x,w} \setminus (N_{u,v} \cup \{v\})| + |N_{u,v,x,w}|) \right] \\
&= \sum_{x \in N_{u,v}} \left[2(P_{u,x} - \mathbb{1}_{\{uv \in E\}} Y_{u,v,x}) - \sum_{w \in N_{u,v,x}} (Y_{u,x,w} - \mathbb{1}_{\{uv \in E\}}) \right] \\
&= \sum_{x \in N_{u,v}} \left[pY_{u,x}^2 \pm 2\delta n^{3/2} p^{3+\hat{\gamma}} \Phi - \sum_{w \in N_{u,v,x}} Y_{u,x,w} + O(np^3) \right] \\
&= \sum_{x \in N_{u,v}} \left[pY_{u,x}^2 \pm 3\delta n^{3/2} p^{3+\hat{\gamma}} \Phi - \sum_{w \in N_{u,v,x}} Y_{u,x,w} \right]. \tag{3.9}
\end{aligned}$$

Putting (3.8) and (3.9) together, taking advantage of the fact that trivially

$$\sum_{xy \in P_{u,v}} 2Y_{u,x,y} = \sum_{x \in N_{u,v}} \sum_{w \in N_{u,v,x}} Y_{u,x,w},$$

we see that the expression in (3.7) is equal to $-1/Q$ times

$$\begin{aligned}
& \sum_{x \in N_{u,v}} \left(pY_{u,x} Y_x \pm 3\delta n^{3/2} p^{3+\hat{\gamma}} \Phi \right) - 2 \left(\sum_{xy \in P_{u,v}} Y_{x,y} \right) \pm (1 + O(\Lambda)) \beta n^{5/2} p^{4+\hat{\gamma}} \Phi \\
&= \sum_{x \in N_{u,v}} \left[pY_{u,x} (np \pm \kappa n^{1/2} p^{\hat{\gamma}-1} \Phi) - Y_{u,v,x} (np^2 \pm \alpha n^{1/2} p^{\hat{\gamma}} \Phi) \right] \\
&\quad \pm \left[(1 + O(\Lambda)) \beta n^{5/2} p^{4+\hat{\gamma}} \Phi + 4\delta n^{5/2} p^{5+\hat{\gamma}} \Phi \right] \\
&= R_{u,v} np^2 \pm \left[(1 + O(\Lambda)) \kappa n^{5/2} p^{4+\hat{\gamma}} \Phi + 6\alpha n^{3/2} p^{2+\hat{\gamma}} \Phi \max\{np^3, \log^5 n\} \right. \\
&\quad \left. + (1 + O(\Lambda)) \beta n^{5/2} p^{4+\hat{\gamma}} \Phi + 4\delta n^{5/2} p^{5+\hat{\gamma}} \Phi \right], \tag{3.10}
\end{aligned}$$

where we used the fact that $R_{u,v} = \sum_{x \in N_{u,v}} (Y_{u,x} - Y_{u,v,x} + \mathbb{1}_{\{uv \in E\}})$ and absorbed the indicator variables (whose total contribution is at most $np^2 Y_{u,v} = O(n^2 p^4)$) into our final error term, as well as the inequality $Y_{u,v,x} \leq 3 \max\{np^3, \log^5 n\}$ which follows from (2.12) by monotonicity.

Now that we have an expression (albeit in 7 parts) for the expected change in $R_{u,v}$, we are ready to consider the expected change in $R_{u,v}$ relative to its expected trajectory. Define

$$X = R_{u,v} - p(Y_u - Y_{u,v})Y_{u,v}$$

and consider $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$. Write

$$\Delta [p(Y_u - Y_{u,v})Y_{u,v}] = p(i+1) \cdot \Delta [(Y_u - Y_{u,v})Y_{u,v}] + \Delta p \cdot (Y_u(i) - Y_{u,v}(i))Y_{u,v}(i). \quad (3.11)$$

We will see that the expected change in $R_{u,v}$ that comes from triangles of types 1-6 will balance off with the first term in (3.11), while the expected change in $R_{u,v}$ due to triangles of type 7 will be balanced with the second term.

Collecting (3.1)-(3.6), the total contribution to $\mathbb{E}[\Delta R_{u,v} \mid \mathcal{F}_i]$ from triangles of types 1-6 equals

$$-\frac{1}{Q} \sum_{x \in N_{u,v}} [Y_{u,x}Y_{v,x} + Y_{u,x}^2 - Y_{u,x}Y_{u,v,x} - 2Y_{v,x}Y_{u,v,x}] - \frac{1}{Q} \sum_{x \in N_u \setminus N_{u,v}} Y_{u,x}Y_{u,v,x} + O\left(\frac{p}{n}\right). \quad (3.12)$$

Furthermore, we can analyze the change in $\Xi = Y_{u,v}(Y_u - Y_{u,v})$ by considering the following 3 cases:

- (i) Selecting a triangle uxy for $x, y \in N_u \setminus N_{u,v}$: the co-neighborhood of u, v is unchanged while $(Y_u - Y_{u,v})$ is decremented by 2, altogether giving $\Delta\Xi = -2Y_{u,v}$.
- (ii) Selecting a triangle vxy for $x \in N_{u,v}$ and $y \neq u$: the co-neighborhood loses x and in addition loses y if $y \in N_{u,v}$, while Y_u remains unchanged. The change in Ξ is thus $2Y_{u,v} - Y_u - 1$ in the former case and $4Y_{u,v} - 2Y_u - 4$ in the latter one.
- (iii) Selecting a triangle uxy for $x \in N_{u,v}$: If $y \in N_{u,v}$ the co-neighborhood loses 2 vertices while $\Delta(Y_u - Y_{u,v}) = 0$, yielding $\Delta\Xi = -2(Y_u - Y_{u,v})$. Similarly, if $y = v$ then $\Delta\Xi = -(Y_u - Y_{u,v})$. Otherwise, $y \in N_u \setminus N_{u,v}$ and both $Y_{u,v}$ and $(Y_u - Y_{u,v})$ decrease by 1, hence $\Delta\Xi = -Y_u + 1$.

Altogether, we can obtain the factors of 2 in Item (i) and in the case $y \in N_{u,v}$ of Items (ii),(iii) automatically by symmetry when summing over x as follows:

$$\begin{aligned} \mathbb{E}[\Delta[Y_{u,v}(Y_u - Y_{u,v})] \mid \mathcal{F}_i] &= -Y_{u,v} \sum_{x \in N_u \setminus N_{u,v}} \frac{Y_{u,x} - Y_{u,v,x}}{Q} \\ &\quad - (Y_u - 2Y_{u,v} - O(1)) \sum_{x \in N_{u,v}} \frac{Y_{v,x} - \mathbb{1}_{\{uv \in E\}}}{Q} \\ &\quad - (Y_u - Y_{u,v}) \left(\sum_{x \in N_{u,v}} \frac{Y_{u,x}}{Q} \right) + ((Y_u - Y_{u,v}) + (-Y_u + 1)) \left(\sum_{y \in N_u \setminus N_{u,v}} \frac{Y_{u,v,y}}{Q} \right). \end{aligned}$$

All the triple-degree terms cancel out and we can sum all the $O(1)$ -terms and rewrite the above as

$$-\frac{1}{Q} \sum_{x \in N_{u,v}} [Y_{u,x}(Y_u - Y_{u,v}) + Y_{v,x}(Y_u - 2Y_{u,v})] - \frac{1}{Q} \sum_{x \in N_u \setminus N_{u,v}} Y_{u,x}Y_{u,v} + O\left(\frac{1}{n}\right).$$

Notice that when multiplying the above by p the error term becomes an additive $O(p/n)$ while the main terms are of order $O(p^2)$. As such, the same estimate holds for the result of multiplying the above by $p(i+1)$ (which differs from $p(i)$ by an additive $O(n^{-2})$ error and thus introduces an extra $O((p/n)^2)$ error term). We can now combine this with the change in $R_{u,v}$ given in (3.12) to get that the contribution to $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ from triangles of types 1-6 and the first term in (3.11) is

$$\frac{1}{Q} \sum_{x \in N_{u,v}} [- (Y_{u,x}Y_{v,x} + Y_{u,x}^2 - Y_{u,x}Y_{u,v,x} - 2Y_{v,x}Y_{u,v,x}) + pY_{u,x}(Y_u - Y_{u,v}) + pY_{v,x}(Y_u - 2Y_{u,v})]$$

$$+\frac{1}{Q} \sum_{x \in N_u \setminus N_{u,v}} [-Y_{u,x}Y_{u,v,x} + pY_{u,v}Y_{u,x}] + O\left(\frac{p}{n}\right),$$

which is equal to

$$\begin{aligned} & -\frac{1}{Q} \sum_{x \in N_{u,v}} (Y_{u,x} + Y_{v,x}) [Y_{u,x} - Y_{u,v,x} - p(Y_u - Y_{u,v})] \\ & -\frac{1}{Q} \sum_{x \in N_u \setminus N_{u,v}} Y_{u,x} (Y_{u,v,x} - pY_{u,v}) + \frac{1}{Q} \sum_{x \in N_{u,v}} Y_{v,x} (Y_{u,v,x} - pY_{u,v}) + O\left(\frac{p}{n}\right). \end{aligned} \quad (3.13)$$

The key observation here is that the first and second summations in (3.13) feature the random variable X itself, a fact which our self-correction argument for X hinges on. Namely, by definition of $R_{u,v}$ we have $\sum_{x \in N_{u,v}} [Y_{u,x} - Y_{u,v,x} - \mathbb{1}_{\{uv \in E\}} - p(Y_u - Y_{u,v})] = R_{u,v} - pY_{u,v}(Y_u - Y_{u,v}) = X$ and similarly $\sum_{x \in N_u \setminus N_{u,v}} [Y_{u,v,x} - pY_{u,v}] = X$, hence we can use our $(1 + O(\Lambda))$ -approximation for the factors $(Y_{u,x} + Y_{v,x})$, $Y_{u,x}$, $Y_{v,x}$ in the summations in (3.13) and obtain that the contribution to $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ from triangles of types 1-6 and the first term in (3.11) is

$$\begin{aligned} & -\frac{12 + O(\Lambda)}{n^2 p} X - \frac{6 + O(\Lambda)}{n^2 p} X + \frac{6 + O(\Lambda)}{n^2 p} \sum_{x \in N_{u,v}} (Y_{u,v,x} - pY_{u,v}) + O\left(\frac{p}{n}\right) \\ & = -\frac{18 + O(\Lambda)}{n^2 p} X + \frac{12 + O(\Lambda)}{n^2 p} \left(P_{u,v} - p \frac{Y_{u,v}^2}{2} \right) + O\left(\frac{p}{n}\right). \end{aligned} \quad (3.14)$$

We now turn to the triangles of type 7. As we noted above, we balance the term (3.10) with the second term in (3.11), i.e. the expected change in $pY_{u,v}(Y_u - Y_{u,v})$ due to the change in p (which deterministically decreases by $6/n^2$). The total contribution to $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ from these terms is

$$\begin{aligned} & -\frac{R_{u,v}np^2}{Q} + \frac{6}{n^2} Y_{u,v}(Y_u - Y_{u,v}) \pm \frac{1}{Q} \left[(1 + O(\Lambda))(\beta + \kappa)n^{5/2}p^{4+\hat{\gamma}}\Phi + 4\delta n^{5/2}p^{5+\hat{\gamma}}\Phi \right. \\ & \quad \left. + 6\alpha n^{3/2}p^{2+\hat{\gamma}}\Phi \max\{np^3, \log^5 n\} \right] \\ & = -\frac{6 + O(\Lambda)}{n^2 p} X \pm (6 + O(\Lambda)) \left[(\beta + \kappa) \frac{p^{1+\hat{\gamma}}\Phi}{n^{1/2}} + (6\alpha + 4\delta) \frac{p^{2+\hat{\gamma}}\Phi}{n^{1/2}} \right], \end{aligned} \quad (3.15)$$

where we used the fact that the term corresponding to taking $\log^5 n$ in $\max\{np^3, \log^5 n\}$ has order $n^{-3}p^{\hat{\gamma}-1}\log^5 n = O(\log^3 n \cdot n^{-1/2}p^{1+\hat{\gamma}}\Phi)$ as long as $p \geq n^{-1/2}\log^4 n$ (which easily holds for large enough n) and hence is negligible compared to the $O(\Lambda)$ -error in the term involving $\beta + \kappa$.

Putting everything together, the combination of (3.14),(3.15) gives

$$\mathbb{E}[\Delta X \mid \mathcal{F}_i] = -\frac{24 + O(\Lambda)}{n^2 p} X \pm (6 + O(\Lambda)) \left[(\beta + \kappa) \frac{p^{1+\hat{\gamma}}\Phi}{n^{1/2}} + (6\alpha + 6\delta) \frac{p^{2+\hat{\gamma}}\Phi}{n^{1/2}} \right] \quad (3.16)$$

and we are ready to establish the concentration of $R_{u,v}$ via a martingale argument. As outlined above, we introduce two critical intervals for the random variable X , corresponding to the upper bound and lower bound on $R_{u,v}$. These intervals have one endpoint at the bound we are trying to establish and the other somewhat closer to zero (corresponding to the expected trajectory of X). For the variable $R_{u,v}$ to violate Eq. (2.7) it must be that X crosses one of the critical intervals.

Our critical interval for the upper bound on $R_{u,v}$ is

$$I_R = \left(\hat{\beta} n^{3/2} p^{2+\hat{\gamma}} \Phi, \beta n^{3/2} p^{2+\hat{\gamma}} \Phi \right) \quad \text{where} \quad \hat{\beta} = (1 - \log^{-1} n) \beta. \quad (3.17)$$

Suppose that $X(i_0)$ enters I_R for the first time at round i_0 (within the time range covered by Theorem 2.1) and define the stopping time $\tau_R = \min\{i \geq i_0 : X(i) < \hat{\beta} n^{3/2} p^{2+\hat{\gamma}} \Phi\}$, i.e. the first time past that point at which X exits the interval through its lower endpoint. We claim that $Z(i \wedge \tau_R)$ is a supermartingale, where

$$Z(i) = X(i) - \beta n^{3/2} p^{2+\hat{\gamma}} \Phi \quad \text{for } i \geq i_0.$$

To see this, write $t = i/n^2$ according to which $p = 1 - 6t$ and $\Phi = e^{6\Upsilon t} \log^2 n$, and note that for any $\hat{\gamma} > 0$ the second derivative of $f(t) = e^{6\Upsilon t} (1 - 6t)^{2+\hat{\gamma}}$ is uniformly bounded in $[0, \frac{1}{6}]$, hence

$$f(t + n^{-2}) = f(t) + \left(6\Upsilon e^{6\Upsilon t} (1 - 6t)^{2+\hat{\gamma}} - 6(2 + \hat{\gamma}) e^{6\Upsilon t} (1 - 6t)^{1+\hat{\gamma}} \right) n^{-2} + O(n^{-4}). \quad (3.18)$$

This provides an estimate for $\Delta[\beta n^{3/2} p^{2+\hat{\gamma}} \Phi]$ between $Z(i+1)$ and $Z(i)$. At the same time, for any $X(i)$ satisfying (3.17) we can plug in the lower bound this gives for X in (3.16) and obtain

$$\begin{aligned} \mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau_R > i] &\leq -(24 + O(\Lambda)) \hat{\beta} \frac{p^{1+\hat{\gamma}} \Phi}{n^{1/2}} + (6 + O(\Lambda)) (\beta + \kappa) \frac{p^{1+\hat{\gamma}} \Phi}{n^{1/2}} \\ &\quad + (36 + O(\Lambda)) (\alpha + \delta) \frac{p^{2+\hat{\gamma}} \Phi}{n^{1/2}} + (1 + O(n^{-2})) \left[6(2 + \hat{\gamma}) \beta \frac{p^{1+\hat{\gamma}} \Phi}{n^{1/2}} - 6\Upsilon \beta \frac{p^{2+\hat{\gamma}} \Phi}{n^{1/2}} \right] \\ &= (-4\hat{\beta} + 3\beta + \kappa + \hat{\gamma}\beta + O(\Lambda)) \frac{6p^{1+\hat{\gamma}} \Phi}{n^{1/2}} + (-\Upsilon\beta + 6\alpha + 6\delta + O(\Lambda)) \frac{6p^{2+\hat{\gamma}} \Phi}{n^{1/2}}. \end{aligned}$$

By the definition of $\hat{\beta}, \hat{\gamma}$ we have $-4\hat{\beta} + 3\beta + \kappa + \hat{\gamma}\beta = -\beta + \kappa + \gamma\beta - 2\beta/\log n$ as the difference between $\hat{\gamma}$ and γ negates the difference between $\hat{\beta}$ and β as well as the $O(\Lambda)$ term, hence we can conclude that if n is large enough then $Z(i \wedge \tau_R)$ is a supermartingale provided that

$$(1 - \gamma)\beta \geq \kappa \quad (3.19)$$

and Υ is sufficiently large, namely $\Upsilon > 6(\alpha + \delta)/\beta$.

To apply Freedman's inequality we need to obtain bounds on $|\Delta Z|$ and $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i]$. Recall that $\Delta Z = \Delta R_{u,v} - \Delta[p\Xi] - \Delta[\beta n^{3/2} p^{2+\hat{\gamma}} \Phi]$ where $\Xi = (Y_u - Y_{u,v})Y_{u,v}$. By (3.11) and the fact that $\Delta p = -6/n^2$ we have $\Delta[p\Xi] = p\Delta\Xi + O(p^3)$, while $\Delta[\beta n^{3/2} p^{2+\hat{\gamma}} \Phi] = O(n^{-1/2} p^{1+\hat{\gamma}} \Phi) = o(1)$. Hence, ΔZ will be dominated by $\Delta R_{u,v} - \Delta[p\Xi]$. There are four cases to consider here:

- (i) Choosing a triangle that includes u or v and some vertex $x \in N_{u,v}$ (triangles of types 1-4,6 in the study of $\mathbb{E}[\Delta R_{u,v} \mid \mathcal{F}_i]$ above): There are $O(n^2 p^4)$ such triangles and selecting one of them affects both $R_{u,v}$ and Ξ . As we next specify, the principle terms in these changes are identical and so $|\Delta Z|$ is bounded by the error terms in our approximations. Indeed, going back to the analysis of the triangle types as well that of $\Delta\Xi$ we recall the various triangle types satisfied:
 - Type 1: $\Delta R_{u,v} = -Y_{u,x} + 2Y_{u,v,x} + O(1)$ vs. $\Delta\Xi = -Y_u + 2Y_{u,v} + O(1)$.
 - Type 2: $\Delta R_{u,v} = -Y_{u,x} + (Y_{u,v,x} - Y_{u,v,y}) + O(1)$ vs. $\Delta\Xi = -Y_u + O(1)$.
 - Type 3: $\Delta R_{u,v} = 2(Y_{u,v,x} + Y_{u,v,y}) - (Y_{u,x} + Y_{u,y}) + O(1)$ vs. $\Delta\Xi = 4Y_{u,v} - 2Y_u + O(1)$.
 - Type 4: $\Delta R_{u,v} = (Y_{u,v,x} + Y_{u,v,y}) - (Y_{u,x} + Y_{u,y}) + O(1)$ vs. $\Delta\Xi = 2Y_{u,v} - 2Y_u$.

- Type 6: $\Delta R_{u,v} = -R_{u,v}$ vs. $\Delta \Xi = Y_{u,v} - Y_u$.

In all of the above cases the main terms of $\Delta R_{u,v}$ cancel those of $p\Delta \Xi$ at the cost of an $O(n^{1/2}p^{\hat{\gamma}}\log^2 n)$ -error due to the approximations of $Y_{u,v}$ and Y_u (dominating all other errors).

- (ii) Choosing a triangle that includes u but no vertex in $N_{u,v}$ (triangles of type 5 above): there are $O(n^2p^3)$ such triangles and each corresponds to $\Delta R_{u,v} = -(Y_{u,v,x} + Y_{u,v,y})$ vs. $\Delta \Xi = -2Y_{u,v}$. Hence, in this case $|\Delta Z| = O(\sqrt{np^3\log^5 n} + \log^5 n)$, with the dominant error-term due to triple co-degrees (exceeding that in $pY_{u,v}$ since $\gamma > \frac{1}{2}$).
- (iii) Choosing triangles that include an edge of $R_{u,v}$ but contain neither u nor v (type 7 triangles): Each of these $O(n^3p^6)$ triangles corresponds to $|\Delta Z| = O(1)$ as $\Delta R_{u,v} = -1$ and $\Delta \Xi = 0$.
- (iv) Choosing any other triangle: as $R_{u,v}$ and Ξ are both unchanged, these triangles can modify Z by at most $O(p^3)$ due to the additive $6/n^2$ change in p .

The L^∞ bound on ΔZ is clearly dominated by rounds of the first sort and $|\Delta Z| = O(n^{1/2}p^{\hat{\gamma}}\log^2 n)$. For an L^2 bound notice that the 4 round types contribute $O(p^{1+2\hat{\gamma}}\log^4 n)$, $O(p^3\log^5 n + \frac{\log^{10} n}{n})$, $O(p^3)$ and $O(p^6)$ respectively to $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i]$. As $\hat{\gamma} < 1$ we have $p^3\log^5 n = o(p^{1+2\hat{\gamma}}\log^5 n)$ while the fact that $p \geq p^*$ (which has order $n^{-1/(4-2\hat{\gamma})+o(1)}$ as was seen in Eq. (2.13)) implies that $\frac{\log^5 n}{n} = o(p^{1+2\hat{\gamma}})$ since $np^{1+2\hat{\gamma}} \geq n^{(3-4\hat{\gamma})/(4-2\hat{\gamma})-o(1)} > n^{1/5}$ for large enough n . Altogether it follows that $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i] = O(p^{1+2\hat{\gamma}}\log^4 n)$. Clearly the L^∞ and the L^2 bounds on ΔZ also hold in the conditional space given $\tau_R > i$.

Recall that we are interested in $Z(i)$ starting at time i_0 , i.e. immediately after X enters the critical interval I_R . Let $p_0 = p(i_0) = 1 - 6i_0/n^2$ and observe that our bound on $|\Delta Z|$ guarantees that $0 \leq X(i_0) - \hat{\beta}n^{3/2}(p_0)^{2\hat{\gamma}}\Phi \leq O(n^{1/2}(p_0)^{\hat{\gamma}}\log^2 n)$. Hence,

$$Z(i_0) \leq (\hat{\beta} - \beta)n^{3/2}(p_0)^{2+\hat{\gamma}}\Phi + O(n^{1/2}(p_0)^{\hat{\gamma}}\log^2 n) \leq -\frac{1}{2}\beta n^{3/2}(p_0)^{2+\hat{\gamma}}\log n,$$

where the final factor of $\frac{1}{2}$ readily cancels the $O(n^{1/2}(p_0)^{\hat{\gamma}}\log^2 n)$ -term for sufficiently large n since $n(p_0)^2\log^{-1} n \geq n^{1-o(1)}(p^*)^2 > n^{1/4}$. We now apply Freedman's inequality (Theorem 3.1) to the supermartingale $S_j = Z((i_0 + j) \wedge \tau_R)$ while noting that the above analysis implies that

$$\begin{aligned} S_0 &\leq -\frac{1}{2}\beta n^{3/2}(p_0)^{2+\hat{\gamma}}\log n, \\ \max_j |S_{j+1} - S_j| &= O\left(n^{1/2}(p_0)^{\hat{\gamma}}\log^2 n\right), \\ \sum_j \mathbb{E}[(S_{j+1} - S_j)^2 \mid \mathcal{F}'_j] &= O\left(\log^5 n \sum_{i \geq i_0} (p(i))^{1+2\hat{\gamma}}\right) = O\left(n^2(p_0)^{2+2\hat{\gamma}}\log^5 n\right), \end{aligned}$$

where $\mathcal{F}'_j = \mathcal{F}_{i_0+j}$. We deduce that for some fixed constant $c > 0$,

$$\mathbb{P}(\cup_j \{S_j \geq 0\}) \leq \exp\left\{-c \frac{n^3(p_0)^{4+2\hat{\gamma}}\log^2 n}{n^2(p_0)^{2+2\hat{\gamma}}\log^5 n + n^2(p_0)^{2+2\hat{\gamma}}\log^3 n}\right\} = \exp\{-cn(p_0)^2\log^{-3} n\},$$

which is sufficiently small to afford a union bound over all u, v and time steps i_0 , hence w.h.p. X never crosses the critical interval I_R and in particular $X(i) \leq \beta n^{3/2}p^{2+\hat{\gamma}}\Phi$ for all u, v and i . Exactly the same argument shows that w.h.p. $X(i) \geq -\beta n^{3/2}p^{2+\hat{\gamma}}\Phi$ for all u, v, i by considering the lower interval $(-\beta n^{3/2}p^{2+\hat{\gamma}}\Phi, -\hat{\beta}n^{3/2}p^{2+\hat{\gamma}}\Phi)$ and analyzing the variable $Z(i) = -X(i) - \beta n^{3/2}p^{2+\hat{\gamma}}\Phi$. This completes the proof of Eq. (2.7),(2.8).

3.2 Edges within the co-neighborhood ($P_{u,v}$)

Our analysis of the variable $P_{u,v}$ (as well as that of the remaining variables throughout this section) will follow the exact same arguments used in Section 3.1 to control $R_{u,v}$, namely we will study the variable $X = P_{u,v} - pY_{u,v}^2$, establish L^1, L^2, L^∞ bounds for its change per round and deduce the result from an appropriate martingale argument. This being the case, we will repeat the arguments only briefly while keeping the main focus on deriving the required estimates on ΔX .

Observe that choosing any triangle incident to some $x \in N_{u,v}$ and either u or v (or both) removes x from the co-neighborhood and thus eliminates $|N_{u,v,x}|$ edges from the tally of $P_{u,v}$. Removing any other triangle xyz (i.e. one which includes neither u nor v) that includes $1 \leq s \leq 3$ edges from the induced subgraph on $N_{u,v}$ will decrease the count in $P_{u,v}$ by these s edges. Overall we get that

$$\begin{aligned} \mathbb{E} [\Delta P_{u,v} \mid \mathcal{F}_i] &= - \sum_{x \in N_{u,v}} \frac{Y_{u,x} + Y_{v,x} - \mathbb{1}_{\{uv \in E\}}}{Q} Y_{u,v,x} - \frac{2P_{u,v}}{Q} - \sum_{xy \in P_{u,v}} \frac{Y_{x,y} - 2}{Q} \\ &= - \sum_{x \in N_{u,v}} \frac{Y_{u,x} + Y_{v,x}}{Q} Y_{u,v,x} - \sum_{xy \in P_{u,v}} \frac{Y_{x,y}}{Q} + O\left(\frac{p^2}{n}\right), \end{aligned} \quad (3.20)$$

where the term $-2P_{u,v}/Q$ is due to the fact that any $xy \in E$ with $x, y \in N_{u,v}$ (a total of $P_{u,v}$ edges) is double-counted in the first summation upon selecting one of the triangles uxy or vxy . Next define

$$X(i) = P_{u,v}(i) - pY_{u,v}^2/2,$$

and note that since $Y_{u,v}(i) - 2 \leq Y_{u,v}(i+1) \leq Y_{u,v}(i)$ we can write $\Delta(Y_{u,v}^2)$ as $(2Y_{u,v} + O(1))\Delta Y_{u,v}$ and using the fact that $p(i+1) = p(i) - 6n^{-2}$ as well as Eq. (2.1) we deduce that

$$\begin{aligned} \mathbb{E} [\Delta (pY_{u,v}^2/2) \mid \mathcal{F}_i] &= -\frac{6}{n^2} \frac{Y_{u,v}^2}{2} + \frac{1}{2} \left(p - \frac{6}{n^2} \right) (2Y_{u,v} + O(1)) \left(- \sum_{x \in N_{u,v}} \frac{Y_{u,x} + Y_{v,x} - \mathbb{1}_{\{uv \in E\}}}{Q} \right) \\ &= -\frac{3Y_{u,v}^2}{n^2} - pY_{u,v} \sum_{x \in N_{u,v}} \frac{Y_{u,x} + Y_{v,x}}{Q} + O\left(\frac{p^2}{n}\right). \end{aligned} \quad (3.21)$$

Combining (3.20), (3.21) and noticing that by definition $\sum_{x \in N_{u,v}} (Y_{u,v,x} - pY_{u,v}) = 2X$ now gives

$$\begin{aligned} \mathbb{E} [\Delta X \mid \mathcal{F}_i] &= -\frac{1}{Q} \sum_{x \in N_{u,v}} (Y_{u,x} + Y_{v,x})(Y_{u,v,x} - pY_{u,v}) - \left(\sum_{xy \in P_{u,v}} \frac{Y_{x,y}}{Q} \right) + \frac{3Y_{u,v}^2}{n^2} + O\left(\frac{p^2}{n}\right) \\ &= -\frac{12 + O(\Lambda)}{n^2 p} \sum_{x \in N_{u,v}} (Y_{u,v,x} - pY_{u,v}) - \frac{np^2 \pm \alpha n^{1/2} p^{\hat{\gamma}} \Phi}{Q} P_{u,v} + \frac{3Y_{u,v}^2}{n^2} + O\left(\frac{p^2}{n}\right) \\ &= -\frac{30 + O(\Lambda)}{n^2 p} X \pm \frac{(3 + O(\Lambda)) \alpha p^{2+\hat{\gamma}} \Phi}{n^{1/2}}, \end{aligned}$$

where we used the approximation that all co-degrees, Q and $P_{u,v}$ are $(1 + O(\Lambda))np^2$, $(\frac{1}{6} + O(\Lambda))n^3 p^3$ and $(\frac{1}{2} + O(\Lambda))n^2 p^5$ respectively at time i by our hypothesis.

Define $\hat{\delta} = (1 - \log^{-1} n)\delta$ and let assume that in some round i_0 the variable $X(i_0)$ enters the upper critical interval given by

$$I_P = \left(\hat{\delta} n^{3/2} p^{3+\hat{\gamma}} \Phi, \delta n^{3/2} p^{3+\hat{\gamma}} \Phi \right) \quad \text{where} \quad \hat{\delta} = (1 - \log^{-1} n) \delta.$$

Let τ_P denote the stopping time $\min\{i \geq i_0 : X(i) < \hat{\delta} n^{3/2} p^{3+\hat{\gamma}} \Phi\}$ and consider the random variable

$$Z(i) = X(i) - \delta n^{3/2} p^{3+\hat{\gamma}} \Phi \quad \text{for } i \geq i_0.$$

Our upper bound on $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ and the fact $|\Delta[\delta n^{3/2} p^{3+\hat{\gamma}} \Phi]| \leq (6 + O(n^{-2}))(3 + \hat{\gamma})\delta n^{-1/2} p^{2+\hat{\gamma}} \Phi$ (derived by the exact same argument as in Eq. (3.18) in Section 3.1) now imply that

$$\mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau_P > i] \leq 3 \left[-10\hat{\delta} + \alpha + (6 + 2\hat{\gamma})\delta + O(\Lambda) \right] \frac{p^{2+\hat{\gamma}} \Phi}{n^{1/2}}.$$

Since the term $2(\gamma - \hat{\gamma}) = 12/\log n$ cancels $10(\delta - \hat{\delta}) = 10/\log n$ as well as the $O(\log^{-2} n)$ term represented by Λ , it follows that $Z(i_0 + j \wedge \tau_P)$ is a supermartingale provided that

$$(4 - 2\gamma)\delta \geq \alpha. \quad (3.22)$$

To obtain L^∞ and L^2 bounds for the one-step changes in Z argue as follows:

- (i) If we choose a triangle that includes an edge joining $\{u, v\}$ with $N_{u,v}$ this then modifies both $P_{u,v}$ and $pY_{u,v}^2/2$ yet by the same leading order terms and hence changes Z by at most $O(\sqrt{np^3 \log^5 n} + \log^5 n)$. The number of these triangles is $O(n^2 p^4)$ as the dominant term here corresponds to edges in $R_{u,v}, S_{u,v}$. Thus, the probability of such an event is at most $O(p/n)$.
- (ii) Selecting a triangle that includes an edge in $N_{u,v}$ but does not include u or v (and hence does not change $Y_{u,v}$) modifies Z by $O(1)$. The asymptotics of $P_{u,v}$ and the co-degrees imply that there are $O(n^3 p^7)$ such triangles and hence the probability of such an event is $O(p^4)$.
- (iii) All other triangles do not change $P_{u,v}, Y_{u,v}$ and hence change Z (via the change in p) by $O(p^4)$.

Overall we get $|\Delta Z| = O(n^{1/2} p^{3/2} \log^{5/2} n + \log^5 n)$ and $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i] = O(p^4 \log^5 n + (p/n) \log^{10} n)$. Letting $p_0 = p(i_0) = 1 - 6i_0/n^2$, the definition of i_0 and our bound on $|\Delta Z|$ guarantee that $0 \leq X(i_0) - \hat{\delta} n^{3/2} (p_0)^{3+\hat{\gamma}} \Phi \leq O(n^{1/2} (p_0)^{\hat{\gamma}} \log^2 n)$. Hence,

$$Z(i_0) \leq (\hat{\delta} - \delta) n^{3/2} (p_0)^{3+\hat{\gamma}} \Phi + O(n^{1/2} (p_0)^{3/2} \log^{5/2} n + \log^5 n) \leq -\frac{1}{2} \delta n^{3/2} (p_0)^{3+\hat{\gamma}} \log n,$$

where the final factor of $\frac{1}{2}$ cancels the $O(n^{1/2} (p_0)^{3/2} \log^{5/2} n + \log^5 n)$ -term for large enough n since $np^{3/2+\hat{\gamma}} \log^{-3/2} n \geq n^{(\frac{5}{2}-3\gamma)/(4-2\gamma)-o(1)} > n^{1/4}$ and $n^{3/2} p^{3+\hat{\gamma}} \log^{-4} n \geq n^{(3-4\gamma)/(4-2\gamma)-o(1)} > n^{1/4}$ for all $p \geq p^*$. We conclude that $S_j = Z((i_0 + j) \wedge \tau_P)$ is a supermartingale with

$$\begin{aligned} S_0 &\leq -\frac{1}{2} \delta n^{3/2} (p_0)^{3+\hat{\gamma}} \log n, \\ \max_j |S_{j+1} - S_j| &= O\left(\sqrt{n(p_0)^3 \log^5 n + \log^5 n}\right), \\ \sum_j \mathbb{E}[(S_{j+1} - S_j)^2 \mid \mathcal{F}'_j] &= O(n^2 (p_0)^5 \log^5 n + n(p_0)^2 \log^{10} n), \end{aligned}$$

where $\mathcal{F}'_j = \mathcal{F}_{i_0+j}$. Recalling that $\hat{\gamma} \geq \frac{1}{2}$ and that in addition $n^{-1/2} \log n = o(p^{2-\hat{\gamma}})$ for all $p \geq p^*$ since $(p^*)^{2-\hat{\gamma}} \geq cn^{-1/2} \log^5 n$ by (2.13), we see that

$$|S_0| \max_j |S_{j+1} - S_j| = O\left(n^2 (p_0)^{\frac{9}{2}+\hat{\gamma}} \log^{\frac{7}{2}} n + n^{\frac{3}{2}} (p_0)^{3+\hat{\gamma}} \log^6 n\right) = o(n^2 (p_0)^5 \log^5 n)$$

and therefore Theorem 3.1 gives that for some fixed $c > 0$,

$$\mathbb{P}(\cup_j \{S_j \geq 0\}) \leq \exp \left\{ -c \frac{n^3 (p_0)^{6+2\hat{\gamma}} \log^2 n}{n^2 (p_0)^5 \log^5 n + n(p_0)^2 \log^{10} n} \right\} = \exp \left\{ -c \frac{n^2 (p_0)^{4+2\hat{\gamma}}}{n(p_0)^3 \log^3 n + \log^8 n} \right\}.$$

Observe that if $p_0 \geq n^{-1/3}$ then the last expression is at most

$$\exp \left\{ -n^{1-o(1)}(p_0)^{1+2\hat{\gamma}} \right\} \leq \exp \left\{ -n^{\frac{2}{3}(1-\hat{\gamma})-o(1)} \right\} = \exp \left\{ -n^{\frac{2}{3}(\sqrt{2}-1)-o(1)} \right\} < e^{-n^{1/4}}$$

for large enough n , and otherwise (recalling that $p_0 \geq p^* \geq n^{-1/(4-2\hat{\gamma})-o(1)}$) it is most

$$\exp \left\{ -n^{2-o(1)}(p_0)^{4+2\hat{\gamma}} \right\} \leq \exp \left\{ -n^{3-\frac{8}{4-2\hat{\gamma}}-o(1)} \right\} = \exp \left\{ -n^{3-2\sqrt{2}-o(1)} \right\} < e^{-n^{1/6}}.$$

In either case, the probability of the event $\cup_j \{S_j \geq 0\}$ is far smaller than any polynomial in n , hence a union bound establishes that w.h.p. $X(i) < \delta n^{3/2} p^{3+\hat{\gamma}} \Phi$ for all u, v and i . A similar argument shows that w.h.p. $X(i) > -\delta n^{3/2} p^{3+\hat{\gamma}} \Phi$ for all u, v and i . This completes the proof of Eq. (2.9).

3.3 Co-degrees $(Y_{u,v})$

Following the arguments in the previous sections we will establish L^1, L^2, L^∞ bounds on the variable

$$X = Y_{u,v} - np^2.$$

As p decreases by $6/n^2$ with each step, Eq. (2.1) and (2.4) show that

$$\begin{aligned} \mathbb{E}[\Delta X \mid \mathcal{F}_i] &= - \sum_{x \in N_{u,v}} \frac{Y_{u,x} + Y_{v,x} - \mathbb{1}_{\{uv \in E\}}}{Q} + \frac{6}{n} (2p - 6/n^2) \\ &= - \frac{4P_{u,v} + R_{u,v} + S_{u,v}}{Q} + \frac{12p}{n} + O\left(\frac{1}{n^2 p}\right), \end{aligned}$$

where the last error term absorbed the indicators and $O(n^{-3})$ from the first expression. Substituting our estimates (2.7),(2.8),(2.9) for $R_{u,v}, S_{u,v}, P_{u,v}$ implies that this is equal to

$$-\frac{1}{Q} \left(p(Y_u + Y_v)Y_{u,v} \pm 4\delta n^{3/2} p^{3+\hat{\gamma}} \Phi \pm 2\beta n^{3/2} p^{2+\hat{\gamma}} \Phi \right) + \frac{12p}{n} + O\left(\frac{1}{n^2 p}\right),$$

and using the estimate (2.10) for Y_u and that $Q = (1 + O(\Lambda))n^3 p^3$ we can conclude that

$$\mathbb{E}[\Delta X \mid \mathcal{F}_i] = - \frac{p(2np \pm 2\kappa n^{1/2} p^{\hat{\gamma}-1} \Phi)Y_{u,v}}{Q} + \frac{12p}{n} \pm (12 + O(\Lambda)) \left[\frac{2\delta p^{\hat{\gamma}} \Phi}{n^{3/2}} + \frac{\beta p^{\hat{\gamma}-1} \Phi}{n^{3/2}} \right] + O\left(\frac{1}{n^2 p}\right).$$

Crucially, we did not approximate Q in the first expression with a $(1 + O(\Lambda))$ correction factor as this would incur an error that would be too large to handle. Instead, there we apply Eq. (2.5) and the fact that $Y_{u,v} = X + np^2$ to obtain that

$$\begin{aligned} \frac{p(2np \pm 2\kappa n^{1/2} p^{\hat{\gamma}-1} \Phi)Y_{u,v}}{Q} &= \frac{12 + O(\Lambda)}{n^2 p} X + \frac{2n^2 p^4}{\frac{1}{6} n^3 p^3 \pm \alpha^2 n^2 p^{2\hat{\gamma}-1} \Phi^2} \pm (12 + O(\Lambda)) \frac{\kappa p^{\hat{\gamma}-1} \Phi}{n^{3/2}} \\ &= \frac{12 + O(\Lambda)}{n^2 p} X + \frac{12p}{n} \left(1 + O\left(\frac{p^{2\hat{\gamma}-4} \Phi^2}{n}\right) \right) \pm (12 + O(\Lambda)) \frac{\kappa p^{\hat{\gamma}-1} \Phi}{n^{3/2}}. \end{aligned}$$

Combining this with the above estimate for $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$, the term $12p/n$ vanishes and we get that

$$\mathbb{E}[\Delta X \mid \mathcal{F}_i] = - \frac{12 + O(\Lambda)}{n^2 p} X \pm (12 + O(\Lambda)) \left[\frac{2\delta p^{\hat{\gamma}} \Phi}{n^{3/2}} + \frac{\beta p^{\hat{\gamma}-1} \Phi}{n^{3/2}} + \frac{\kappa p^{\hat{\gamma}-1} \Phi}{n^{3/2}} \right] + O\left(\frac{p^{2\hat{\gamma}-3} \Phi^2}{n^2}\right),$$

where the $O(1/(n^2p))$ error term was absorbed into the $O(n^{-2}p^{2\hat{\gamma}-3}\Phi^2)$ -term since $\hat{\gamma} \leq 1$ and so $1/(n^2p) = o(p^{2\hat{\gamma}-3}\Phi^2/n^2)$. Furthermore, we claim that one may now omit this latter error-term altogether as it is negligible compared to the error-term of $O(\Lambda)$ in the terms involving β, κ . Indeed, keeping in mind that Φ and Λ^{-1} are each of order $\log^2 n$, we have

$$\frac{p^{2\hat{\gamma}-3}\Phi^2}{n^2} = O(\Lambda) \frac{p^{\hat{\gamma}-1}\Phi}{n^{3/2}} \frac{\log^4 n}{n^{1/2}p^{2-\hat{\gamma}}} \leq O(\Lambda) \frac{p^{\hat{\gamma}-1}\Phi}{n^{3/2}} \frac{\log^4 n}{\sqrt{n(p^*)^{4-2\hat{\gamma}}}} = O(\Lambda) \frac{p^{\hat{\gamma}-1}\Phi \log^4 n}{n^{3/2} \log^5 n} = o\left(\Lambda \frac{p^{\hat{\gamma}-1}\Phi}{n^{3/2}}\right). \quad (3.23)$$

Assume now that $X(i)$ enters the upper critical interval defined by

$$I_Y = \left(\hat{\alpha} n^{1/2} p^{\hat{\gamma}} \Phi, \alpha n^{1/2} p^{\hat{\gamma}} \Phi\right) \quad \text{where} \quad \hat{\alpha} = (1 - \log^{-1} n) \alpha.$$

That is, suppose that i_0 is the first round at which $X(i) \geq \hat{\alpha} n^{1/2} p^{\hat{\gamma}} \Phi$ and define the stopping time $\tau_Y = \min\{i \geq i_0 : X(i) < \hat{\alpha} n^{1/2} p^{\hat{\gamma}} \Phi\}$. As before, consider

$$Z(i) = X(i) - \alpha n^{1/2} p^{\hat{\gamma}} \Phi.$$

By the same argument of (3.18) we have

$$\Delta \left[\alpha n^{1/2} p^{\hat{\gamma}} \Phi \right] = (6 + O(n^{-2})) \left(\Upsilon p^{\hat{\gamma}} - \hat{\gamma} p^{\hat{\gamma}-1} \right) \alpha n^{-3/2} \Phi$$

and combined this with the above upper bound on $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ establishes that

$$\begin{aligned} \mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau_Y > i] &\leq [-2\hat{\alpha} + 2\beta + 2\kappa + \hat{\gamma}\alpha + O(\Lambda)] \frac{6p^{\hat{\gamma}-1}\Phi}{n^{3/2}} + [4\delta - \Upsilon\alpha + O(\Lambda)] \frac{6p^{\hat{\gamma}}\Phi}{n^{3/2}} \\ &\leq [-(2-\gamma)\alpha + 2\beta + 2\kappa] \frac{6p^{\hat{\gamma}-1}\Phi}{n^{3/2}} + [4\delta - \Upsilon\alpha + O(\Lambda)] \frac{6p^{\hat{\gamma}}\Phi}{n^{3/2}}, \end{aligned}$$

where we used the fact that $\gamma - \hat{\gamma} = 6/\log n$ to absorb both $2(\alpha - \hat{\alpha}) = 2/\log n$ and the $O(\Lambda)$ -term for large n . In particular, $S_j = Z((i_0 + j) \wedge \tau_Y)$ is indeed a supermartingale so long as

$$(2 - \gamma)\alpha \geq 2\beta + 2\kappa \quad (3.24)$$

and Υ sufficiently large, namely $\Upsilon > 4\delta/\alpha$.

It remains to bound ΔZ in L^∞ and L^2 . Here there are 2 types of rounds: ones in which we choose a triangle that involves u or v and a vertex in $Y_{u,v}$ (there are $O(n^2p^4)$ such triangles) and ones where we choose any other triangle, in which case $Y_{u,v}$ is unchanged. The former event has probability $O(p/n)$ and leads to an $O(1)$ change in Z while the latter gives a variation in Z of order $O(p/n)$ due to the $-6/n^2$ change in p . Therefore, $|\Delta Z| = O(1)$ and $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i] = O(p/n)$.

Let $p_0 = p(i_0) = 1 - 6i_0/n^2$. By the definition of i_0 and the fact that $|\Delta Z| = O(1)$,

$$Z(i_0) \leq (\hat{\alpha} - \alpha) n^{1/2} (p_0)^{\hat{\gamma}} \Phi + O(1) \leq -\frac{1}{2} \alpha n^{1/2} (p_0)^{\hat{\gamma}} \log n$$

(the last inequality holds for large n as the final expression clearly tends to ∞ with n), and therefore the supermartingale $S_j = Z((i_0 + j) \wedge \tau_P)$ satisfies

$$S_0 \leq -\frac{1}{2} \alpha n^{1/2} (p_0)^{\hat{\gamma}} \log n,$$

$$\begin{aligned} \max_j |S_{j+1} - S_j| &= O(1), \\ \sum_j \mathbb{E}[(S_{j+1} - S_j)^2 \mid \mathcal{F}'_j] &= O(n(p_0)^2), \end{aligned}$$

where $\mathcal{F}'_j = \mathcal{F}_{i_0+j}$. Since $n^{1/2}(p_0)^{2-\hat{\gamma}} \geq c \log^5 n$ for $p_0 \geq p^*$ due to Eq. (2.13) it follows that $|S_0| \max_j |S_{j+1} - S_j| = O(n(p_0)^2)$ and so Theorem 3.1 yields that for some fixed $c > 0$

$$\mathbb{P}(\cup_j \{S_j \geq 0\}) \leq \exp \left\{ -c \frac{n(p_0)^{2\hat{\gamma}} \log^2 n}{n(p_0)^2} \right\} = \left\{ -c(p_0)^{2\hat{\gamma}-2} \log^2 n \right\} \leq e^{-c \log^2 n},$$

which is sufficiently small to show that w.h.p. $X(i) < \alpha n^{1/2} p^{\hat{\gamma}} \Phi$ for all u, v and i . The same argument handles the analogous symmetric case of the critical interval $(-\alpha n^{1/2} p^{\hat{\gamma}} \Phi, -\hat{\alpha} n^{1/2} p^{\hat{\gamma}} \Phi)$ and shows that w.h.p. $X(i) > -\alpha n^{1/2} p^{\hat{\gamma}} \Phi$ for all u, v and i . This concludes the proof of Eq. (2.6).

3.4 Edges within a neighborhood (T_u)

The number of edges in the subgraph induced by the neighborhood of u can change in two ways: Either a vertex is removed from N_u (due to selecting a triangle of the form uxy with $x \in N_u$) thereby decrementing T_u by all edges incident to it in this induced subgraph, or N_u remains unchanged (upon selecting a triangle that does not include u) and yet some of its inner edges are removed.

The former case will be handled by directly summing over $x \in N_u$, noting there are Y_{ux} triangles of the form uxy whereas the vertex x is incident to Y_{ux} edges counted in T_u . The latter case requires a more delicate treatment, similar to the one used to study $R_{u,v}$ in Section 3.1. Indeed, the naive approach would be to sum over edges counted by T_u , i.e. $xy \in E$ with $x, y \in N_u$, as each of these would decrease T_u by 1 upon selecting one of the $Y_{x,y}$ triangles incident to it. However, the cumulative error in this approach (summing the co-degree errors for each edge in T_u) would be quite substantial as it completely ignores the effect of averaging the co-degrees over T_u . To take advantage of this point we will use our estimates for the random variables $S_{u,x}$ and $P_{u,x}$, which incorporate this averaging effect. Namely, we have

$$\mathbb{E}[\Delta T_u \mid \mathcal{F}_i] = -\frac{1}{Q} \sum_{x \in N_u} Y_{u,x}^2 + \frac{T_u}{Q} - \frac{1}{Q} \sum_{x \in N_u} P_{u,x} - \frac{1}{2Q} \sum_{x \in N_u} S_{u,x}, \quad (3.25)$$

where the first two terms account for triangles of the form uxy (the correction term T_u/Q is due to the double-count by $x, y \in N_u$), the third term accounts for triangles of the form xyz where $x, y, z \in N_u$ and the final term accounts for triangles of the form xyz where $x, y \in N_u$ and $z \notin N_u$.

We evaluate the last two terms in (3.25) using the bounds (2.8),(2.9) for $S_{u,v}$ and $P_{u,v}$ to get

$$\begin{aligned} -\frac{1}{2Q} \sum_{x \in N_u} (2P_{u,x} + S_{u,x}) &= -\frac{1}{2Q} \sum_{x \in N_u} \left(pY_{u,x}^2 + pY_{x,u}(Y_x - Y_{x,u}) \pm \beta n^{3/2} p^{2+\hat{\gamma}} \Phi \pm 2\delta n^{3/2} p^{3+\hat{\gamma}} \Phi \right) \\ &= -\frac{p}{2Q} \sum_{x \in N_u} (Y_x Y_{x,u}) \pm (1 + O(\Lambda)) \left[\frac{3\beta p^{\hat{\gamma}} \Phi}{n^{1/2}} + \frac{6\delta p^{1+\hat{\gamma}} \Phi}{n^{1/2}} \right]. \end{aligned} \quad (3.26)$$

We now need the following simple lemma.

Lemma 3.2. *Let $a_1, \dots, a_m \in \mathbb{R}$ and suppose that $|a - a_i| \leq \varepsilon$ for all i and some $a \in \mathbb{R}$. Then*

$$\frac{(\sum_i a_i)^2}{m} \leq \sum_i a_i^2 \leq \frac{(\sum_i a_i)^2}{m} + 4m\varepsilon^2.$$

Proof. The lower bound is due to Cauchy-Schwarz. For the upper bound fix $b = \frac{1}{m} \sum_i a_i$ and note that the assumption on the a_i 's implies that $|b - a_i| \leq 2\varepsilon$ for all i . Observe that the convex function $\sum_i a_i^2$ achieves its maximum over the convex set $\{\sum_i a_i = bm\} \cap (\bigcap_i \{|b - a_i| \leq 2\varepsilon\})$ at an extremal point where $a_i = b \pm 2\varepsilon$ for all i . Hence, $a_i = b - 2\varepsilon$ for $\lfloor m/2 \rfloor$ indices, $a_i = b + 2\varepsilon$ for $\lfloor m/2 \rfloor$ indices and if m is odd there is a single $a_i = b$. In particular

$$\sum_{i=1}^m a_i^2 \leq \lfloor m/2 \rfloor (b - 2\varepsilon)^2 + \lfloor m/2 \rfloor (b + 2\varepsilon)^2 + b^2 \mathbb{1}_{\{m \equiv 1 \pmod{2}\}} \leq mb^2 + 4m\varepsilon^2. \quad \square$$

Revisiting Eq. (3.25), apply the above lemma with $m = Y_u = (1 + O(\Lambda))np$ and $\varepsilon = \alpha n^{1/2} p^{\hat{\gamma}} \Phi$ to estimate the first sum there (noting that $\sum_{x \in N_u} Y_{ux} = 2T_u$) and then use (3.26) to get

$$\begin{aligned} \mathbb{E}[\Delta T_u \mid \mathcal{F}_i] &= -\frac{1}{Q} \left(\frac{4T_u^2}{Y_u} \pm (5 + O(\Lambda)) \alpha^2 n^2 p^{1+2\hat{\gamma}} \Phi^2 \right) - \frac{p}{2Q} \sum_{x \in N_u} (Y_x Y_{x,u}) \\ &\quad \pm (1 + O(\Lambda)) \left[\frac{3\beta p^{\hat{\gamma}} \Phi}{n^{1/2}} + \frac{6\delta p^{1+\hat{\gamma}} \Phi}{n^{1/2}} \right] + O(1/n). \end{aligned} \quad (3.27)$$

As usual set

$$X(i) = T_u - pY_u^2/2$$

and consider $\Delta(pY_u^2/2)$. Observe that Y_u changes if and only if the triangle selected is of the form uxy with $x, y \in N_u$, in which case it decreases by 2. Hence, $\mathbb{E}[\Delta Y_u \mid \mathcal{F}_i] = -2T_u/Q$ and $\Delta(Y_u^2) = (2Y_u - 2)\Delta Y_u$, and putting these together we get

$$\mathbb{E}[\Delta(pY_u^2/2) \mid \mathcal{F}_i] = -\frac{6}{n^2} \frac{Y_u^2}{2} - \frac{1}{2} \left(p - \frac{6}{n^2} \right) (2Y_u - 2) \frac{2T_u}{Q} = -\frac{3Y_u^2}{n^2} - \frac{2pT_u Y_u}{Q} + O\left(\frac{p}{n}\right).$$

Combining this estimate with (3.27) and the bound (2.10) for Y_u gives

$$\begin{aligned} \mathbb{E}[\Delta X \mid \mathcal{F}_i] &= -\frac{4T_u}{QY_u} \left(T_u - p \frac{Y_u^2}{2} \right) - \frac{p}{2Q} \left(\sum_{x \in N_u} Y_{x,u} \left(np \pm \kappa n^{1/2} p^{\hat{\gamma}-1} \Phi \right) \right) + \frac{3Y_u^2}{n^2} \\ &\quad \pm (1 + O(\Lambda)) \left[\frac{3\beta p^{\hat{\gamma}} \Phi}{n^{1/2}} + \frac{6\delta p^{1+\hat{\gamma}} \Phi}{n^{1/2}} + \frac{30\alpha^2 p^{2\hat{\gamma}-2} \Phi^2}{n} \right] + O\left(\frac{p}{n}\right). \end{aligned}$$

By Eq. (3.23) we have $n^{-1} p^{2\hat{\gamma}-2} \Phi^2 = o(\Lambda n^{-1/2} p^{\hat{\gamma}} \Phi)$ for all $p \geq p^*$, thus the above expression involving α^2 can be absorbed into the $O(\Lambda)$ error-term of the expression involving β . Furthermore, since $4T_u/(QY_u) = (12 + O(\Lambda))/(n^2 p)$ and

$$\frac{p}{2Q} \left(\sum_{x \in N_u} Y_{x,u} \left(np \pm \kappa n^{1/2} p^{\hat{\gamma}-1} \Phi \right) \right) - \frac{3Y_u^2}{n^2} = \frac{6 + O(\Lambda)}{n^2 p} (T_u - pY_u^2/2) \pm (3 + O(\Lambda)) \frac{\kappa p^{\hat{\gamma}} \Phi}{n^{1/2}}$$

we can conclude that

$$\mathbb{E}[\Delta X \mid \mathcal{F}_i] = -\frac{18 + O(\Lambda)}{n^2 p} X \pm (1 + O(\Lambda)) \left[\frac{3(\beta + \kappa) p^{\hat{\gamma}} \Phi}{n^{1/2}} + \frac{6\delta p^{1+\hat{\gamma}} \Phi}{n^{1/2}} \right].$$

Now we consider the upper critical interval for T_u given by

$$I_T = \left(\hat{\mu} n^{3/2} p^{1+\hat{\gamma}} \Phi, \mu n^{3/2} p^{1+\hat{\gamma}} \Phi \right) \quad \text{where} \quad \hat{\mu} = (1 - \log^{-1} n) \mu,$$

and as before let i_0 be the first round in which $X(i) \geq \hat{\mu} n^{3/2} p^{1+\hat{\gamma}} \Phi$, define the stopping time $\tau_T = \min\{i \geq i_0 : X(i) < \hat{\mu} n^{3/2} p^{1+\hat{\gamma}} \Phi\}$ and consider

$$Z(i) = X(i) - \mu n^{3/2} p^{1+\hat{\gamma}} \Phi.$$

Exactly the argument of (3.18) gives

$$\Delta \left[\mu n^{3/2} p^{1+\hat{\gamma}} \Phi \right] = (6 + O(n^{-2})) \left(\Upsilon p^{1+\hat{\gamma}} - (1 + \hat{\gamma}) p^{\hat{\gamma}} \right) \mu n^{-1/2} \Phi$$

and together with the above upper bound on $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ we get

$$\begin{aligned} \mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau_T > i] &\leq [-6\hat{\mu} + \beta + \kappa + (2 + 2\hat{\gamma})\mu + O(\Lambda)] \frac{3p^{\hat{\gamma}}\Phi}{n^{1/2}} + [\delta - \Upsilon\mu + O(\Lambda)] \frac{6p^{1+\hat{\gamma}}\Phi}{n^{1/2}} \\ &\leq [-(4 - 2\gamma)\mu + \beta + \kappa] \frac{3p^{\hat{\gamma}}\Phi}{n^{1/2}} + [\delta - \Upsilon\mu + O(\Lambda)] \frac{6p^{1+\hat{\gamma}}\Phi}{n^{1/2}}, \end{aligned}$$

where we used the fact that $\gamma - \hat{\gamma} = 6/\log n$ to absorb both $4(\mu - \hat{\mu}) = 4/\log n$ and the $O(\Lambda)$ -term for large n . In particular, $S_j = Z((i_0 + j) \wedge \tau_T)$ is indeed a supermartingale provided that

$$(4 - 2\gamma)\mu \geq \beta + \kappa \quad (3.28)$$

and Υ is sufficiently large, namely $\Upsilon > \delta/\mu$.

Having established an L^1 bound on ΔS it remains to consider the corresponding L^2, L^∞ bounds. If we choose a triangle of the form uxy , an event that has probability $O(1/n)$, then T_u decrease by $Y_{ux} + Y_{uy} - 1$ while Y_u decreases by 2, hence the change in Z in this case is at most the $O(n^{1/2} p^{\hat{\gamma}} \log^2 n)$ due to the error-terms in our approximation for the degrees and co-degrees. The probability that we choose a triangle that does not contain u yet includes an edge in T_u is $O(p^2)$ and selecting such a triangle changes Z by $O(1)$. The choice of any other triangle changes Z by $O(p^2)$ due to the change in p . Altogether, $|\Delta Z| = O(n^{1/2} p^{\hat{\gamma}} \log^2 n)$ and $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i] = O(p^{2\hat{\gamma}} \log^4 n)$. Let $p_0 = p(i_0) = 1 - 6i_0/n^2$ and recall that the definition of i_0 and our bound on $|\Delta Z|$ ensure that

$$Z(i_0) \leq (\hat{\mu} - \mu) n^{3/2} (p_0)^{1+\hat{\gamma}} \Phi + O(n^{1/2} (p_0)^{\hat{\gamma}} \log^2 n) \leq -\frac{1}{2} \mu n^{3/2} (p_0)^{1+\hat{\gamma}} \log n,$$

where the factor of $\frac{1}{2}$ absorbs the $O(n^{1/2} (p_0)^{\hat{\gamma}} \log^2 n)$ -term since $np_0 \log^{-1} n \geq n^{1-o(1)} p^* > \sqrt{n}$ for large enough n . It then follows the supermartingale $S_j = Z((i_0 + j) \wedge \tau_T)$ satisfies

$$\begin{aligned} S_0 &\leq -\frac{1}{2} \mu n^{3/2} (p_0)^{1+\hat{\gamma}} \log n, \\ \max_j |S_{j+1} - S_j| &= O\left(n^{1/2} p^{\hat{\gamma}} \log^2 n\right), \\ \sum_j \mathbb{E}[(S_{j+1} - S_j)^2 \mid \mathcal{F}'_j] &= O\left(n^2 (p_0)^{1+2\hat{\gamma}} \log^4 n\right), \end{aligned}$$

where $\mathcal{F}'_j = \mathcal{F}_{i_0+j}$. Here $|S_0| \max_j |S_{j+1} - S_j| = o(n^2 (p_0)^{1+2\hat{\gamma}} \log^4 n)$ due to one extra log factor between these expressions and therefore Theorem 3.1 establishes that for some fixed $c > 0$

$$\mathbb{P}(\cup_j \{S_j \geq 0\}) \leq \exp \left\{ -c \frac{n^3 (p_0)^{2+2\hat{\gamma}} \log^2 n}{n^2 (p_0)^{1+2\hat{\gamma}} \log^4 n} \right\} = \{-cnp_0 \log^{-2} n\} \leq e^{-\sqrt{n}}.$$

We conclude that w.h.p. $X(i) < \mu n^{3/2} p^{1+\hat{\gamma}} \Phi$ for all u and i , and the same argument shows that w.h.p. $X(i) > -\mu n^{3/2} p^{\hat{\gamma}} \Phi$ for all u and i . This concludes the proof of Eq. (2.11).

3.5 Vertex degrees (Y_u)

The analysis of the degrees will be straightforward using our estimate (2.11) for T_u , the number of inner edges in the neighborhood of a vertex u , since Y_u changes iff the triangle selected is of the form uxy (in which case it decreases by 2). Indeed, setting

$$X(i) = Y_u - np,$$

our bounds on T_u and Q imply that

$$\begin{aligned} \mathbb{E}[\Delta X \mid \mathcal{F}_i] &= -\frac{2T_u}{Q} + \frac{6}{n} = -\frac{pY_u^2 \pm 2\mu n^{3/2}p^{1+\hat{\gamma}}\Phi}{\frac{1}{6}n^3p^3 \pm \alpha^2 n^2 p^{2\hat{\gamma}-1}\Phi^2} + \frac{6}{n} \\ &= -\frac{p(X^2 + (np)^2 + 2npX)}{\frac{1}{6}n^3p^3} + \frac{6}{n} \pm (12 + O(\Lambda)) \frac{\mu p^{\hat{\gamma}-2}\Phi}{n^{3/2}} + O\left(\frac{p^{2\hat{\gamma}-4}\Phi^2}{n^2}\right) \\ &= -\frac{12}{n^2p}X \pm (12 + O(\Lambda)) \frac{\mu p^{\hat{\gamma}-2}\Phi}{n^{3/2}}, \end{aligned}$$

where the term $pX^2/(n^3p^3)$ was absorbed into the $O(p^{2\hat{\gamma}-4}\Phi^2/n^2)$ -term since $|X| \leq \kappa\sqrt{np}^{\hat{\gamma}-1}\Phi$ by (2.10), and this latter error-term was thereafter omitted as it is $o(\Lambda p^{\hat{\gamma}-2}\Phi/n^{3/2})$ by (3.23).

Now consider the upper critical interval for Y_u given by

$$I'_Y = \left(\hat{\kappa}n^{1/2}p^{\hat{\gamma}-1}\Phi, \kappa n^{1/2}p^{\hat{\gamma}-1}\Phi \right) \quad \text{where} \quad \kappa = (1 - \log^{-1} n) \kappa,$$

and as before assume that i_0 is the first time at which $X(i_0) \geq \hat{\kappa}n^{1/2}p^{\hat{\gamma}-1}\Phi$ and define the stopping time $\tau'_Y = \min\{i > i_0 : X(i) < \hat{\kappa}n^{1/2}p^{\hat{\gamma}-1}\Phi\}$. With these definitions, the variation in the variable

$$Z(i) = X(i) - \kappa n^{1/2}p^{\hat{\gamma}-1}\Phi$$

consists of ΔX as well as $\Delta[-\kappa n^{1/2}p^{\hat{\gamma}-1}\Phi] \leq (6 + O(n^{-2}))(\hat{\gamma} - 1)\kappa p^{\hat{\gamma}-2}n^{-3/2}\Phi + O(n^{-7/2})$, hence

$$\mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau'_Y > i] \leq [-2\hat{\kappa} + 2\mu + (\hat{\gamma} - 1)\kappa + O(\Lambda)] \frac{6p^{\hat{\gamma}-2}\Phi}{n^{3/2}} \leq [-(3 - \gamma)\kappa + 2\mu] \frac{6p^{\hat{\gamma}-2}\Phi}{n^{3/2}},$$

where we used the fact that $\gamma - \hat{\gamma} = 6/\log n$ eliminates the term $2(\kappa - \hat{\kappa}) = 2/\log n$ and $O(\Lambda)$ -term for large n . Hence, $S_j = Z((i_0 + j) \wedge \tau'_Y)$ is a supermartingale as long as

$$(3 - \gamma)\kappa \geq 2\mu. \quad (3.29)$$

Furthermore, in each round we either select a triangle incident to u , an event which has probability $O(1/n)$ and changes Z by $O(1)$, or we do not affect Y_u and thus change Z by $O(1/n)$ due to the change in p . Thus, $|\Delta Z| = O(1)$ while $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i] = O(1/n)$, and we conclude that for large enough n the supermartingale S_j has the following attributes:

$$S_0 \leq -\frac{1}{2}\kappa n^{1/2}(p_0)^{\hat{\gamma}-1} \log n, \quad \max_j |S_{j+1} - S_j| = O(1), \quad \sum_j \mathbb{E}[(S_{j+1} - S_j)^2 \mid \mathcal{F}'_j] = O(np_0),$$

where $p_0 = p(i_0) = 1 - 6i_0/n^2$ and $\mathcal{F}'_j = \mathcal{F}_{i_0+j}$. Since $\sqrt{n}(p_0)^{2-\hat{\gamma}} \geq \sqrt{n}(p^*)^{2-\hat{\gamma}} \geq c \log^5 n$ we deduce that $|S_0| \max_j |S_{j+1} - S_j| = o(np_0)$ and thus Theorem 3.1 establishes that for some fixed $c > 0$

$$\mathbb{P}(\cup_j \{S_j \geq 0\}) \leq \exp \left\{ -c \frac{n(p_0)^{2\hat{\gamma}-2} \log^2 n}{np_0} \right\} = \left\{ -cp_0^{2\hat{\gamma}-3} \log^2 n \right\} \leq e^{-c \log^2 n}.$$

Altogether, w.h.p. $X(i) < \kappa n^{1/2}p^{\hat{\gamma}-1}\Phi$ for all u and i , and similarly $X(i) > -\kappa n^{1/2}p^{\hat{\gamma}-1}\Phi$ w.h.p. for all u and i . This completes the proof of Eq. (2.10).

3.6 Co-degree of triples $(Y_{u,v,w})$

We will prove the following result from which (2.12) will readily follow:

$$Y_{u,v,w} = np^3 \pm \sqrt{np^3 \log^5 n} \quad \text{for all } u, v, w \text{ and } p \geq p_1 := n^{-1/3} \log^{5/3} n. \quad (3.30)$$

Indeed, since $np^3 \geq \sqrt{np^3 \log^5 n}$ for all $p \geq p_1$ the above would imply that $Y_{u,v,w} \leq 2np^3$ for all $p \geq p_1$. Since $Y_{u,v,w}$ is monotone decreasing along our process whereas $2np_1^3 = \log^5 n$ it would then follow that $Y_{u,v,w} \leq \max\{np^3 + \sqrt{np^3 \log^5 n}, 2np_1^3\}$ for all p , as required.

It remains to prove (3.30). Define

$$X(i) = Y_{u,v,w}(i) - np^3.$$

We have

$$\begin{aligned} \mathbb{E}[\Delta X \mid \mathcal{F}_i] &= - \sum_{x \in N_{u,v,w}} \frac{Y_{x,u} + Y_{x,v} + Y_{x,w} - \mathbb{1}_{\{uv \in E\}} - \mathbb{1}_{\{uw \in E\}} - \mathbb{1}_{\{vw \in E\}}}{Q} + \frac{18p^2}{n} \\ &\leq - \frac{(X + np^3)(3np^2 - \alpha n^{1/2} p^{\hat{\gamma}} \Phi)}{n^3 p^3 / 6} + O\left(\frac{\alpha^2 p^{2\hat{\gamma}-2} \Phi^2}{n^2}\right) + \frac{18p^2}{n} \\ &= - \frac{18}{n^2 p} X + \frac{6\alpha p^{\hat{\gamma}} \Phi}{n^{3/2}} + O\left(\frac{\alpha p^{\hat{\gamma}-3/2} \Phi \log^2 n}{n^2}\right) \end{aligned}$$

Now suppose that i_0 is a first round at which $X(i_0) > \frac{2}{3} \sqrt{np^3 \log^5 n}$, that is X enters the critical interval

$$I_Y'' = \left(\frac{2}{3} \sqrt{np^3 \log^5 n}, \sqrt{np^3 \log^5 n} \right).$$

Setting $\tau_Y'' = \min\{i > i_0 : X(i) < \frac{2}{3} \sqrt{np^3 \log^5 n}\}$ and

$$Z(i) = X(i) - \sqrt{np^3 \log^5 n}$$

we get

$$\begin{aligned} \mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau_Y'' > i] &\leq - \frac{12p^{1/2} \log^{5/2} n}{n^{3/2}} + \frac{6\alpha p^{\hat{\gamma}} \Phi}{n^{3/2}} + O\left(\frac{\alpha p^{\hat{\gamma}-3/2} \Phi \log^2 n}{n^2}\right) + \frac{9p^{1/2} \log^{5/2} n}{n^{3/2}} \\ &= \frac{6\alpha p^{\hat{\gamma}} \Phi - 3p^{1/2} \log^{5/2} n}{n^{3/2}} + O\left(\frac{\alpha p^{\hat{\gamma}-3/2} \Phi \log^2 n}{n^2}\right). \end{aligned}$$

Since $\hat{\gamma} = 2 - \sqrt{2} - o(1) > \frac{1}{2}$ for large n , it follows that $p^{\hat{\gamma}} \Phi = O(p^{1/2} \log^2 n) = o(p^{1/2} \log^{5/2} n)$ and hence the first term in the above r.h.s. is equal to $-(3 + o(1))(p^{\frac{\log^5 n}{n^3}})^{1/2}$. As for the second term there, recall that $p \geq p^*$ and so by (2.13) we have $p^{4-2\hat{\gamma}} \geq 6\alpha^2 \Phi^2 n^{-1} \log^6 n$. In particular,

$$\frac{p^{\hat{\gamma}-3/2} \Phi \log^2 n}{n^2} = \left(\frac{p \log^5 n}{n^3}\right)^{1/2} \frac{\Phi}{n^{1/2} p^{2-\hat{\gamma}} \log^{1/2} n} \leq \left(\frac{p \log^5 n}{n^3}\right)^{1/2} \frac{1}{\sqrt{6\alpha} \log^{7/2} n},$$

and altogether we conclude that

$$\mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau_Y'' > i] \leq -(3 + o(1)) \frac{p^{1/2} \log^{5/2} n}{n^{3/2}} < 0,$$

where the last inequality holds any for sufficiently large n and confirms that $S_j = Z((i_0 + j) \wedge \tau_Y'')$ is a supermartingale. Moreover, the equation that specified $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ shows that if the triangle selected goes through u, v or w and a vertex in $N_{u,v,w}$, which happens with probability $O(p^2/n)$, then the change in Z is $O(1)$, and otherwise the change in Z is $O(p^2/n)$. Hence, $|\Delta Z| = O(1)$ while $\mathbb{E}[(\Delta Z)^2 \mid \mathcal{F}_i] = O(p^2/n)$, thus the supermartingale S_j has the following attributes:

$$S_0 \leq -\frac{1}{4}\sqrt{n(p_0)^3 \log^5 n}, \quad \max_j |S_{j+1} - S_j| = O(1), \quad \sum_j \mathbb{E}[(S_{j+1} - S_j)^2 \mid \mathcal{F}_j'] = O(n(p_0)^3),$$

where the factor $\frac{1}{4}$ in the first expression (as opposed to $\frac{1}{3}$) treated the potential $O(1)$ deviation of $Z(i_0)$ from the lower endpoint of the critical interval. Noting that $p_0 \geq p_1 = (\frac{\log^5 n}{n})^{1/3}$ and hence $|S_0| \max_j |S_{j+1} - S_j| = O(n(p_0)^3)$, an application of Theorem 3.1 yields that for some fixed $c > 0$,

$$\mathbb{P}(\cup_j \{S_j \geq 0\}) \leq \exp \left\{ -c \frac{n(p_0)^3 \log^5 n}{n(p_0)^3} \right\} = \exp(-c \log^5 n).$$

By the usual union bound over vertices and rounds we now conclude that w.h.p. $X(i) < \sqrt{np^3 \log^5 n}$ for all u, v, w and i , and similarly $X(i) > -\sqrt{np^3 \log^5 n}$ w.h.p. for all u, v, w and i , thus completing the proof of Eq. (2.12).

3.7 Number of triangles (Q)

In the companion paper [5] it is shown (see Theorem 2 there) that w.h.p.

$$Q(i) \leq \frac{n^3 p^3}{6} + \frac{1}{3} n^2 p$$

throughout the process, hence it only remains to prove the lower bound in (2.5). Let

$$X(i) = Q - \frac{n^3 p^3}{6}$$

and recall that due to (2.2) we have

$$\mathbb{E}[\Delta Q \mid \mathcal{F}_i] = - \sum_{xyz \in Q} \frac{Y_{x,y} + Y_{x,z} + Y_{y,z} - 2}{Q} \geq -\frac{1}{Q} \sum_{xy \in E} Y_{x,y}^2.$$

To bound $\mathbb{E}[\Delta X \mid \mathcal{F}_i]$ from below we will thus need an upper bound on $\sum_{xy \in E} Y_{x,y}^2$. Recall that $\sum_{xy \in E} Y_{x,y} = 3Q$ and that $Y_{x,y} = np^2 \pm \alpha n^{1/2} p^{\hat{\gamma}} \Phi$ by Eq. (2.6), hence we can apply Lemma 3.2 together with the fact that $|E(i)| = n^2 p/2 - n/2$ to obtain that

$$\begin{aligned} \frac{1}{Q} \sum_{xy \in E} Y_{x,y}^2 &\leq \frac{1}{Q} \left(\frac{9Q^2}{|E|} + 4|E| \alpha^2 n p^{2\hat{\gamma}} \Phi^2 \right) \leq \frac{18Q}{n^2 p (1 - \frac{1}{np})} + \frac{2n^2 p}{(\frac{1}{6} + O(\Lambda)) n^3 p^3} \alpha^2 n p^{2\hat{\gamma}} \Phi^2 \\ &\leq \frac{18Q}{n^2 p} + O(p) + (12 + O(\Lambda)) \alpha^2 p^{2\hat{\gamma}-2} \Phi^2 = \frac{18Q}{n^2 p} + (12 + O(\Lambda)) \alpha^2 p^{2\hat{\gamma}-2} \Phi^2, \end{aligned}$$

where in the last equality we absorbed the $O(p)$ -term into the $O(\Lambda)$ error-term factor of the last expression since the facts $\hat{\gamma} \leq 1$ and $\Lambda \Phi^2 \geq c \log^3 n$ imply that $p^{2-2\hat{\gamma}} = o(\Lambda \Phi^2)$, i.e. $\Lambda p^{2\hat{\gamma}-2} \Phi^2 \rightarrow \infty$. Adding this to our estimate for $\mathbb{E}[\Delta Q \mid \mathcal{F}_i]$ while observing that $\Delta(-\frac{1}{6} n^3 p^3) = 3np^2 + O(p/n)$ yields

$$\mathbb{E}[\Delta X \mid \mathcal{F}_i] \geq -\frac{18Q}{n^2 p} - (12 + O(\Lambda)) \alpha^2 p^{2\hat{\gamma}-2} \Phi^2 + 3np^2 + O(p/n).$$

As before we incorporate the $O(p/n)$ term into the $O(\Lambda)$ error and using the definition of X we can then rewrite the above as an upper bound on $\Delta(-X)$, as follows:

$$\mathbb{E}[\Delta(-X) \mid \mathcal{F}_i] \leq \frac{18}{n^2 p} X + (12 + O(\Lambda)) \alpha^2 p^{2\hat{\gamma}-2} \Phi^2.$$

Now assume that i_0 is the first round where X drops below $-\hat{\alpha}^2 n^2 p^{2\hat{\gamma}-1} \Phi^2$, i.e. enters the interval

$$I_Q = \left(-\alpha^2 n^2 p^{2\hat{\gamma}-1} \Phi^2, -\hat{\alpha}^2 n^2 p^{2\hat{\gamma}-1} \Phi^2 \right) \quad \text{where} \quad \hat{\alpha} = (1 - \log^{-1} n)^{1/2} \alpha.$$

Further let $\tau_Q = \min\{i > i_0 : X(i) > -\hat{\alpha}^2 n^2 p^{2\hat{\gamma}-1} \Phi^2\}$ and

$$Z(i) = -X(i) - \alpha^2 n^2 p^{2\hat{\gamma}-1} \Phi^2.$$

Since $\Delta[-\alpha^2 n^2 p^{2\hat{\gamma}-1} \Phi^2] \leq (6 + O(n^{-2})) (2\hat{\gamma} - 1) \alpha^2 p^{2\hat{\gamma}-2} \Phi^2$, the upper bound on $\Delta(-X)$ gives

$$\mathbb{E}[\Delta Z \mid \mathcal{F}_i, \tau_Q > i] \leq [-3\hat{\alpha}^2 + 2\alpha^2 + (2\hat{\gamma} - 1)\alpha^2 + O(\Lambda)] 6p^{2\hat{\gamma}-2} \Phi^2 \leq -12(1 - \gamma) \alpha^2 p^{2\hat{\gamma}-2} \Phi^2,$$

where the last inequality used the term $\gamma - \hat{\gamma} = 6/\log n$ to both cancel the $O(\Lambda)$ -term and replace $3\hat{\alpha}^2$ by $3\alpha^2$. As $\gamma < 1$ we deduce that $S_j = Z((i_0 + j) \wedge \tau_Q)$ is indeed a supermartingale.

Next consider the one-step variation of Z . Denoting the selected triangle in a given round by xyz , the change in Q following this round is at most $Y_{x,y} + Y_{x,z} + Y_{y,z}$ and in light of our co-degree estimate (2.6) this expression deviates from its expected value of $3np^2$ by at most $3\alpha n^{1/2} p^{\hat{\gamma}} \log^2 n$. In particular, $|\Delta Z| = O(\sqrt{n}(p_0)^{\hat{\gamma}} \log^2 n)$ and letting $p_0 = p(i_0) = 1 - 6i_0/n^2$ this ensures that

$$Z(i_0) \leq (\hat{\alpha}^2 - \alpha^2) n^2 (p_0)^{2\hat{\gamma}-1} \Phi^2 + O(\sqrt{n}(p_0)^{\hat{\gamma}} \log^2 n) \leq -\frac{1}{2} \alpha^2 n^2 (p_0)^{2\hat{\gamma}-1} \log^3 n,$$

where the last inequality holds for large n since $n^{3/2}(p_0)^{\hat{\gamma}-1} \log n$ tends to ∞ with n . Recalling that there at most $n^2 p_0$ steps remaining until the process terminates, Hoeffding's inequality establishes that for some fixed $c > 0$,

$$\mathbb{P}(\cup_j \{S_j \geq 0\}) \leq \exp \left\{ -c \frac{(n^2(p_0)^{2\hat{\gamma}-1} \log^3 n)^2}{n^2 p_0 (\sqrt{n}(p_0)^{\hat{\gamma}} \log^2 n)^2} \right\} = \exp \left\{ -c n p_0^{2\hat{\gamma}-3} \log^2 n \right\} \leq e^{-n}.$$

We conclude that w.h.p. $X(i) > -\alpha^2 n^2 p^{2\hat{\gamma}-1} \Phi^2$ for all i , completing the proof of Eq. (2.5).

This completes the proof of Theorem 2.1. □

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